**Day 142** **A Level Maths: S Tier 10-a-day: Model Solutions**

## 1 Poisson approximation to a binomial (CIE)

X ~ B(100, 0.02).
(a) Explain why a Poisson approximation is suitable and state its distribution.
(b) Use it to estimate P(X > 3), to 4 s.f.
(c) Find the exact value of P(X > 3), to 4 s.f.

n = 100 is large and p = 0.02 is small, with np = 2
X is approximately Po(2)
$P(X > 3) \approx 1 - P(Y \leq 3) = 0.1429$
Exact: $1 - P(X \leq 3)$ with X ~ B(100, 0.02) = 0.1410
**Answer: (a) Po(2) (b) 0.1429 (c) 0.1410**

## 2 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The sum of the first n terms of a sequence is
$S_n = 2n^2 + n$.
(a) Find an expression for $u_n$.
(b) Find the least n for which $u_n > 162$.

$u_n = S_n - S_{n-1} = 2n^2 + n - [2(n-1)^2 + (n-1)]$
$= 4n - 1$.
$u_1 = S_1 = 3$ also fits, so $u_n = 4n - 1$ for all n.
$4n - 1 > 162 \Rightarrow n > \frac{163}{4}$, so n = 41.
**Answer: (a) $u_n = 4n - 1$ (b) n = 41**

## 3 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 6 cm, subtends an angle of $\frac{2\pi}{3}$ at O.
Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 6^2 \times \frac{2\pi}{3}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = 9\sqrt{3}$
Area = $12\pi - 9\sqrt{3}$
Chord AB = $2r\sin(\theta/2) = 6\sqrt{3}$
Arc AB = $r\theta = 4\pi$
Perimeter = arc + chord
**Answer: Area = $12\pi - 9\sqrt{3}$ cm$^2$, perimeter = $4\pi + 6\sqrt{3}$ cm**

## 4 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(-4\mathbf{i} + 4\mathbf{j})$ N, $(5\mathbf{i} + k\mathbf{j})$ N and $(4\mathbf{i} + 3\mathbf{j})$ N act on a particle of mass 5 kg. The resultant is parallel to $(5\mathbf{i} - \mathbf{j})$.
Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(5)\mathbf{i} + (7 + k)\mathbf{j}$
Parallel to $(5\mathbf{i} - \mathbf{j})$: $\frac{7 + k}{5} = \frac{-1}{5}$, so k = −8
R = $(5\mathbf{i} - \mathbf{j})$ N, so $|R| = \sqrt{(5^2 + -1^2)} = 5.099$
a = |R| ÷ 5
**Answer: (a) k = −8 (b) 1.02 m s$^{-2}$**

## 5 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = x^2 - 4$. Also $fg(x) = -3x^2 + 16$, where f is a linear function. (a) Find f(x).
(b) Solve $gf(x) = 165$.

(a) Let $f(x) = ax + b$. $fg(x) = a(x^2 - 4) + b = -3x^2 + 16$.
So $a = -3$ and $12 + b = 16$, giving $b = 4$.
$f(x) = -3x + 4$.
(b) $gf(x) = (-3x + 4)^2 - 4 = 165$.
$(-3x + 4)^2 = 169$, so $-3x + 4 = \pm 13$.
$x = -3$ or $x = \frac{17}{3}$.

**Answer: (a) $f(x) = -3x + 4$ (b) $x = -3$ or $x = \frac{17}{3}$**

## 6 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 4 red, 6 blue and 3 green counters. A player takes two without replacement. Two reds win £5, two blues win £7, anything else wins nothing.
(a) Find the expected winnings.
(b) The player pays £1 to play. Find the expected profit.

$P(RR) = \frac{4}{13} \times \frac{3}{12} = \frac{1}{13}$

$P(BB) = \frac{6}{13} \times \frac{5}{12} = \frac{5}{26}$

(a) $E = 5 \times \frac{1}{13} + 7 \times \frac{5}{26} = \frac{45}{26}$

(b) Expected profit $= \frac{45}{26} - 1 = \frac{19}{26}$

**Answer: (a) £1.73 (b) £0.731**

## 7 Car on a hill: power and maximum speed (CIE)

A car of mass 1000 kg, with engine power 7.9 kW, climbs a straight road with $\sin\alpha = \frac{1}{20}$ against a constant resistance of 300 N.
Find (a) the greatest steady speed up the hill, (b) the acceleration at 2.5 m s$^{-1}$. Take $g = 9.8$ m s$^{-2}$.

Steady speed: driving force $= 300 + 1000 \times 9.8 \times \frac{1}{20} = 790$ N

$P = F v$, so $v = \frac{7900}{790} = 10$

At 2.5 m s$^{-1}$: $F = \frac{7900}{2.5} = 3160$ N

$F - 790 = 1000\,a$, so $a = \frac{2370}{1000}$

**Answer: (a) 10 m s$^{-1}$ (b) 2.37 m s$^{-2}$**

## 8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{1}{20 + 8\sin\theta + 15\cos\theta}$

(a) Write $8\sin\theta + 15\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

(b) Find the greatest and least values of $f(\theta)$, and the smallest positive $\theta$ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{8^2 + 15^2} = 17$, $\tan\alpha = \frac{15}{8}$, $\alpha = 61.9°$

Denominator $= 20 + 17\sin(\theta + 61.9°)$, which lies between 3 and 37

Greatest $f(\theta) = \frac{1}{3} = \frac{1}{3}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{1}{37} = \frac{1}{37}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 61.9° = 270°$, so $\theta = 208.1°$

**Answer: (a) $17\sin(\theta + 61.9°)$ (b) greatest $\frac{1}{3}$, least $\frac{1}{37}$, $\theta = 208.1°$**

## 9 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x - y = -1$

$2x^2 - 2xy + y^2 = 37$

From the first equation $x = -1 + y$.

Substitute into the second equation and collect terms:

this gives $y^2 - 2y - 35 = 0$.

$y = -5$ or $y = 7$, giving $x = -6$ or $x = 6$.

**Answer: (x, y) = (−6, −5) and (6, 7)**

## 10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two drones P and Q move so that, t seconds after launch, their position vectors in metres are

$\mathbf{r}_P = \begin{pmatrix}-4\\2\\-3\end{pmatrix} + t\begin{pmatrix}0\\1\\-2\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}-11\\-1\\10\end{pmatrix} + t\begin{pmatrix}2\\2\\-5\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-7\\-3\\13\end{pmatrix} + t\begin{pmatrix}2\\1\\-3\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}2\\1\\-3\end{pmatrix} = 0$

$-56 + t(14) = 0$, so $t = 4$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}1\\1\\1\end{pmatrix}$, with length $\sqrt{3}$

**Answer: t = 4, minimum distance = $\sqrt{3}$ m**