Day 141 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n(n + 6)(n + 12)(n + 18) + 1296$ is a perfect square for every integer n.

Pair the factors: $n(n + 18) = n^2 + 18n$ and $(n + 6)(n + 12) = n^2 + 18n + 72$.

Let $u = n^2 + 18n$. Expression $= u(u + 72) + 1296$

$= u^2 + 72u + 1296 = (u + 36)^2$

$= (n^2 + 18n + 36)^2$, a perfect square.

Answer: Proved

2 Dropped and projected particles meet (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Ball A is projected vertically upwards from ground level with speed 31 m s^{-1}. At the same instant, ball B is released from rest 62 m directly above A.

The balls collide. Find (a) the height above the ground at which they collide, (b) the velocity of A just before the collision. Take $g = 9.8$ m s^{-2}.

Relative to B, A moves upwards at the constant speed 31 m s^{-1} (both have acceleration g)

So the balls meet when $31t = 62$, giving $t = 2$

Height of A: $s = 31 \times 2 - \frac{1}{2} \times 9.8 \times 2^2 = 42.4$

$v = u - gt = 31 - 9.8 \times 2 = 11.4$

Answer: (a) 42.4 m (b) 11.4 m s^{-1} upwards

3 Conditional probability from given values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For events A and B, $P(A) = 0.65$, $P(B) = 0.6$ and $P(A \mid B) = 0.9$.

(a) Find $P(A \cap B)$.

(b) Find $P(A' \mid B')$.

$P(A \cap B) = P(A \mid B) \times P(B) = 0.9 \times 0.6 = 0.54$

$P(A \cup B) = 0.65 + 0.6 - 0.54 = 0.71$

$P(B') = 0.4$

$P(A' \cap B') = 1 - P(A \cup B) = 0.29$

$P(A' \mid B') = \frac{0.29}{0.4}$

Answer: (a) 0.54 (b) $\frac{29}{40}$

4 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The volume of a sphere increases at a constant rate of 24π $cm^3 s^{-1}$.

When the radius is 2 cm, find the rate of increase of (a) the radius,

(b) the surface area. Volume $= \frac{4}{3}\pi r^3$.

$dV/dt = 4\pi r^2 \, dr/dt$, so $24\pi = 4\pi \times 4 \times dr/dt$

$dr/dt = \frac{3}{2}$

$A = 4\pi r^2$, so $dA/dt = 8\pi r \, dr/dt = 8\pi \times 2 \times \frac{3}{2}$

Answer: (a) $\frac{3}{2}$ cm s^{-1} (b) 24π $cm^2 s^{-1}$

5 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ax^n$. The line of best fit for lg y against lg x is $\lg y = 0.75 - 0.5 \lg x$.

(a) Find a and n, giving a to 3 significant figures.

(b) Estimate y when $x = 100$, to 3 significant figures.

$y = ax^n$ gives $\lg y = \lg a + n \lg x$, so the gradient is n and the intercept is lg a

$n = -0.5$

$\lg a = 0.75$ so $a = 10^{0.75} = 5.623$

$x = 100$: $y = 5.623 \times 100^{-0.5}$

$= 0.562$

Answer: (a) a = 5.62, n = −0.5 (b) 0.562

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6 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{4}{x^2 + 5}$ at the point where $x = 1$ meets the axes at A and B. Find the exact area of triangle OAB, where O is the origin.

At $x = 1$: $y = \frac{2}{3}$. Differentiate: gradient $= -\frac{2}{9}$

Tangent: $y - \frac{2}{3} = (-\frac{2}{9})(x - 1)$, so $y = -\frac{2}{9}x + \frac{8}{9}$

y-intercept $\frac{8}{9}$; x-intercept 4

Area $= \frac{1}{2} \times |\frac{8}{9}| \times |4|$

Answer: $\mathbf{\frac{16}{9}}$

7 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic $y = f(x)$, which touches the x-axis once. Find $f(x)$, giving your answer in expanded form.

Crosses at $x = 4$ and touches at $x = -3$, so $f(x) = k(x - 4)(x + 3)^2$.

At (0, 36): $36 = k(-4)(3)^2 = -36k$, so $k = -1$.

$f(x) = -(x - 4)(x + 3)^2$, then expand.

Answer: $\mathbf{f(x) = -x^3 - 2x^2 + 15x + 36}$

8 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A shell is projected from a point O on horizontal ground with speed 63.7 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{5}{12}$. Find the horizontal distance between the two points at which the shell is 25.725 m above the ground. Take $g = 9.8$ m s^{-2}.

Vertical: $25.725 = 24.5\,t - 4.9t^2$, i.e. $4.9t^2 - 24.5t + 25.725 = 0$

$t = \frac{24.5 \pm \sqrt{(24.5^2 - 19.6 \times 25.725)}}{9.8} = \frac{24.5 \pm 9.8}{9.8}$

The two times differ by 2 s (symmetry about the top of the path)

Horizontal distance $= 58.8 \times 2$

Answer: 117.6 m

9 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{2}{(x + 2)} > \frac{4}{(x - 2)}$.

Multiply by the positive quantity $(x + 2)^2(x - 2)^2$:

$2(x - (-2))(x - 2)^2 > 4(x - 2)(x - (-2))^2$.

Move to one side and factorise: $(x - (-2))(x - 2)(-2x - 12) > 0$.

Critical values: -2, 2, -6. Test the four regions.

Answer: $\mathbf{x < -6}$ **or** $\mathbf{-2 < x < 2}$

10 Volumes of revolution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The region bounded by $y = 2\sqrt{\sin x}$, the x-axis and the lines $x = 0$ and $x = \pi$ is rotated through 2π about the x-axis. Find the exact volume of the solid formed.

$V = \pi\int_0^{\pi} y^2\,dx = \pi\int_0^{\pi} (2\sqrt{\sin x})^2\,dx$

Simplify the integrand, then integrate

$V = \pi \times (8)$

Answer: $\mathbf{8\pi}$