Day 140 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Compound change with logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drug dose of 80 mg decays by 12% each hour.
Find the first whole number of hours after which less than 5 mg remains.

$80 \times 0.88^n < 5$, so $0.88^n < \frac{1}{16}$

$n > \frac{\ln \frac{1}{16}}{\ln 0.88} = 21.69$ ($\ln 0.88$ is negative, so the inequality reverses)

$n = 22$

Answer: 22

2 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{12}{x^2 - 2x + 2}$ for all real x.

Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$x^2 - 2x + 2 = (x - 1)^2 + 1$.

$(x - 1)^2 \geq 0$, so the denominator is at least 1, with equality at $x = 1$.

The denominator is always positive, so $f(x) > 0$.

Greatest $f = \frac{12}{1} = 12$ at $x = 1$.

Answer: greatest value 12 at x = 1; range $0 < f(x) \leq 12$

3 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $e^x = 4x + 8$ has a root α.
(a) Show that the equation can be written as $x = \ln(4x + 8)$.
(b) Use $x_{n+1} = \ln(4x_n + 8)$ with $x_0 = 3$ to find x_1, x_2, x_3
to 4 d.p., and hence state α to 3 d.p.

Take natural logarithms of both sides to obtain $x = \ln(4x + 8)$

$x_1 = 2.9957$, $x_2 = 2.9949$, $x_3 = 2.9947$

The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.995$

Answer: $x_1 = 2.9957$, $x_2 = 2.9949$, $x_3 = 2.9947$; $\alpha = 2.995$

4 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sqrt{6}$ is irrational, prove by contradiction that $\sqrt{2} + \sqrt{3}$ is irrational.

Assume $\sqrt{2} + \sqrt{3} = r$, where r is rational.

Square both sides: $2 + 3 + 2\sqrt{6} = r^2$.

So $\sqrt{6} = \frac{r^2 - 5}{2}$.

The right-hand side is rational, but $\sqrt{6}$ is irrational.

This is a contradiction, so $\sqrt{2} + \sqrt{3}$ is irrational.

Answer: Proved

5 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(\mathbf{i} + 37\mathbf{j})$ m s^{-1} and $(9\mathbf{i} + 22\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).
Both are still in flight after 2 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $1t\mathbf{i} + (37t - 4.9t^2)\mathbf{j}$; position of B: $9t\mathbf{i} + (22t - 4.9t^2)\mathbf{j}$

Subtracting, the $4.9t^2$ terms cancel: $\mathrm{AB} = t(-8\mathbf{i} + 15\mathbf{j})$

After 2 s: $\mathrm{AB} = (-16\mathbf{i} + 30\mathbf{j})$

Distance $= \sqrt{(16^2 + 30^2)} = 34$

The direction of AB is that of $(-8\mathbf{i} + 15\mathbf{j})$, which does not depend on t: tan angle $= \frac{15}{8}$

Answer: (a) 34 m (b) 61.9°

6 Stratified sample then probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school has 200 students in Year 10, 80 in Year 11 and 120 in the sixth form.
A stratified sample of 40 students is taken.
(a) Find the number sampled from each group.
(b) Two students are chosen at random from the sample. Find the probability they are from the same group.

Sample sizes: $40 \times \frac{200}{400}$, $40 \times \frac{80}{400}$, $40 \times \frac{120}{400} = 20, 8, 12$

P(same) = $20 \times 19 + 8 \times 7 + 12 \times 11/40 \times 39$

$= \frac{568}{1560} = 0.364$

Answer: (a) 20, 8, 12 (b) 0.364

7 Least force to drag a box (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box of weight 78 N is pulled at constant speed along rough horizontal ground ($\mu = \frac{5}{12}$) by a force P N at angle θ above the horizontal.
(a) Find P when $\tan\theta = \frac{12}{5}$. (b) Find the least possible value of P as θ varies, and the corresponding θ to 1 d.p.

Vertical: $R + P\sin\theta = 78$. Horizontal: $P\cos\theta = \mu R = \frac{5}{12}(78 - P\sin\theta)$

$P(\cos\theta + \frac{5}{12}\sin\theta) = \frac{5}{12} \times 78$, so $P = 32.5 \div (\cos\theta + \frac{5}{12}\sin\theta)$

(a) $\cos\theta = \frac{5}{13}$, $\sin\theta = \frac{12}{13}$: $P = 32.5 \div \frac{10}{13} = 42.25$

(b) P is least when $\cos\theta + \frac{5}{12}\sin\theta$ is greatest. Its greatest value is $\sqrt{(1 + \frac{5}{12}^2)} = \frac{13}{12}$,

which occurs when $\tan\theta = \frac{5}{12}$. Then $P = 32.5 \div \frac{13}{12} = 30$ and $\theta = 22.6°$

Answer: (a) 42.25 N (b) 30 N at θ = 22.6°

8 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Write $\frac{2x + 11}{(x + 4)^2}$ in the form $\frac{A}{x + 4} + \frac{B}{(x + 4)^2}$.

(b) Find the exact value of $\int_0^2 \frac{2x + 11}{(x + 4)^2}\,dx$.

$2x + 11 \equiv A(x + 4) + B$, so A = 2, and B = 3

Integral = $[2\ln(x + 4) - \frac{3}{x + 4}]$

Substitute limits 0 and 2

Answer: (a) A = 2, B = 3; (b) $\frac{1}{4} - 2\ln 2 + 2\ln 3$

9 Alternate turns and sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B take turns, A first, to score; the first to score wins. P(A scores) = $\frac{1}{6}$, P(B scores) = $\frac{1}{3}$ each attempt.
Find P(A wins (a) on attempt 1 or 2, (b) overall).

P(A wins at attempt 1) = $\frac{1}{6}$; to reach attempt 2 both miss first: $(\frac{5}{6})(\frac{2}{3}) = \frac{5}{9}$

(a) $\frac{1}{6} + \frac{5}{9} \times \frac{1}{6} = \frac{7}{27}$

(b) P(A wins) = $\frac{1}{6} + \frac{5}{9} \times \frac{1}{6} + \frac{5}{9}^2 \times \frac{1}{6} + \ldots$

Geometric: first term $\frac{1}{6}$, common ratio $\frac{5}{9}$

$S = \frac{a}{1 - r} = \frac{\frac{1}{6}}{\frac{4}{9}} = \frac{3}{8}$

Answer: (a) $\frac{7}{27}$ (b) $\frac{3}{8}$

10 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{2x - 4}{(1 + 2x)(1 - 3x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $2x - 4 \equiv A(1 - 3x) + B(1 + 2x)$: $A = -2$, $B = -2$.

$f(x) = \frac{-2}{1 + 2x} - \frac{2}{1 - 3x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use $c = 2$, then $c = -3$).

Coefficients of x^j: $-2(-2)^j - 2(3)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) $A = -2$, $B = -2$ (b) $-4 - 2x - 26x^2 - 38x^3 + \ldots$; $|x| < \frac{1}{3}$