Day 14 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Force up a rough slope: find weight and friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight W N is on a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{3}{5}$. A force P N acts up a line of greatest slope; the particle stays at rest for $43.2 \leq P \leq 100.8$.
Find (a) W, (b) the coefficient of friction, (c) the acceleration when P = 201.6. Take g = 9.8 m s^{-2}.

Limiting friction acts down the plane at P = 100.8 and up the plane at P = 43.2
$P_{max} = W\sin\alpha + F$ and $P_{min} = W\sin\alpha - F$, where $F = \mu W\cos\alpha$

Adding: $2W\sin\alpha = 144$, so $W = \frac{144}{2 \times \frac{3}{5}} = 120$

Subtracting: $2F = 57.6$, so $F = 28.8 = \mu \times 120 \times \frac{4}{5}$, giving $\mu = 0.3$

P = 201.6: net force up the plane = 201.6 − 100.8 = 100.8, and mass = $\frac{120}{9.8}$

Answer: (a) 120 N (b) 0.3 (c) 8.23 m s^{-2}

2 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 34 cm and its area is 66 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 34$ and $\frac{1}{2}r^2\theta = 66$

$r\theta = 34 - 2r$, so $\frac{1}{2}r(34 - 2r) = 66$

$r^2 - 17r + 66 = 0$, so $(r - 6)(r - 11) = 0$

r = 6: $r\theta = 22$, $\theta = \frac{11}{3}$ r = 11: $r\theta = 12$, $\theta = \frac{12}{11}$

Answer: r = 6, $\theta = \frac{11}{3}$ or r = 11, $\theta = \frac{12}{11}$

3 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $(n + 6)^3 - n^3 - 216$ is a multiple of 18 for every integer n.

$(n + 6)^3 = n^3 + 18n^2 + 108n + 216$.
So the expression = $18n^2 + 108n$
$= 18n(n + 6)$, a multiple of 18.
Answer: Proved

4 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^x + 8e^{-x} = 6$,
giving your answers exactly in terms of natural logarithms.

Multiply by e^x: $e^{2x} - 6e^x + 8 = 0$
$(e^x - 2)(e^x - 4) = 0$
$e^x = 2$ or $e^x = 4$
Answer: x = ln 2 and x = ln 4

5 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(25 - x)^{-\frac{1}{2}}$ up to and including the term in x^2.

(b) Use x = 0.04 to estimate $\frac{1}{\sqrt{24.96}}$ to 5 decimal places.

(a) $(25 - x)^{-\frac{1}{2}} = \frac{1}{5}(1 - \frac{1}{25}x)^{-\frac{1}{2}}$.

$= \frac{1}{5} + \frac{1}{250}x + \frac{3}{25000}x^2 + ...$

(b) 25 − x with x = 0.04 gives 24.96.

Substitute: $\frac{1}{5} + \frac{1}{250}(0.04) + \frac{3}{25000}(0.04)^2$.

= 0.20016 (5 d.p.).

Answer: (a) $\frac{1}{5} + \frac{1}{250}x + \frac{3}{25000}x^2$ (b) 0.20016

6 Screening test and Bayes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A faulty component is present in 0.5% of a population. A test is positive for 98% of those with it and for 5% of those without it.

Find P(positive) and P(has it | positive).

P(positive) = 0.005 × 0.98 + 0.995 × 0.05

= 0.0049 + 0.04975 = 0.05465

P(has it | positive) = $\frac{0.0049}{0.05465}$

= 0.08966

Answer: (a) 0.05465 (b) 0.0897

7 Equidistant point and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A(−3, −10) and B(3, 8). The point C lies on the x-axis and CA = CB. Find the coordinates of C and the area of triangle ABC.

C = (c, 0): $(c - (-3))^2 + 100 = (c - 3)^2 + 64$.

Expand and solve: 12c = −36, so c = −3.

Area = ½|$(x_A - c)y_B - (x_B - c)y_A$|

= ½|0 × 8 − 6 × (−10)| = 30.

Answer: C(−3, 0); area 30

8 Trigonometric integrals with identities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use a suitable trigonometric identity to find the exact value of

$\int_0^{\pi/4} \sin 5x \cos x \, dx$

Use 2 sin A cos B ≡ sin(A + B) + sin(A − B)

Integrate term by term

Substitute the limits

Answer: $\frac{1}{3}$

9 Car on a hill: power and maximum speed (CIE)

A car of mass 1500 kg, with engine power 51.75 kW, climbs a straight road with $\sin\alpha = \frac{1}{10}$ against a constant resistance of 600 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at $\frac{25}{3}$ m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force = $600 + 1500 \times 9.8 \times \frac{1}{10} = 2070$ N

$P = F v$, so $v = \frac{51750}{2070} = 25$

At $\frac{25}{3}$ m s^{-1}: $F = \frac{51750}{\frac{25}{3}} = 6210$ N

$F - 2070 = 1500\,a$, so $a = \frac{4140}{1500}$

Answer: (a) 25 m s^{-1} (b) 2.76 m s^{-2}

10 Sum of two independent copies of a discrete variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X takes values 2, 3, 4, 5 with $P(X = x) = k(x + 1)$. X_1 and X_2 are independent copies of X.

(a) Find k.

(b) Find $P(X_1 + X_2 = 7)$.

(c) Find $P(X_1 = X_2)$ and $E(X_1 + X_2)$.

(a) $\Sigma P = k(3 + 4 + 5 + 6) = 18k = 1$, so $k = \frac{1}{18}$

(b) Ways to total 7: (2,5), (3,4), (4,3), (5,2)

$P = \frac{1}{6}\times\frac{1}{3} + \frac{2}{9}\times\frac{5}{18} + \frac{5}{18}\times\frac{2}{9} + \frac{1}{3}\times\frac{1}{6} = \frac{19}{81}$

(c) $P(X_1 = X_2) = \Sigma [P(X = x)]^2 = \frac{43}{162}$

$E(X) = \frac{34}{9}$, so $E(X_1 + X_2) = 2 \times \frac{34}{9} = \frac{68}{9}$

Answer: (a) $k = \frac{1}{18}$ (b) $\frac{19}{81}$ (c) $\frac{43}{162}$, $\frac{68}{9}$