Day 139 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = 2x^2 + 20x + 56$.

Complete the square: $y = 2(x + 5)^2 + 6$.

Step 1: $y = x^2 \rightarrow y = 2x^2$: a stretch, scale factor 2, parallel to the y-axis.

Step 2: replace x by (x + 5) and add 6: translation by $\begin{pmatrix} -5 \\ 6 \end{pmatrix}$.

Answer: a stretch, scale factor 2, parallel to the y-axis, then translation $\begin{pmatrix} -5 \\ 6 \end{pmatrix}$

2 Histogram: frequency density (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The histogram shows the times, t seconds, of some runners.

Find (a) the total number of runners, (b) an estimate of the number with $15 \leq t < 41$.

Frequency = frequency density × width. Bar 28 to 34: height 2 squares, width 6, frequency 12

So one square of frequency density = 1 runners per second

Class frequencies = 84, 20, 12, 12, 50

Total = 178

Estimate = 1 × [5(1) + 1(12) + 2(6) + 5(7)] = 64

Answer: (a) 178 (b) 64

3 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(4 - x)^{-\frac{1}{2}}$ up to and including the term in x^2.

(b) Use x = 0.04 to estimate $\frac{1}{\sqrt{3.96}}$ to 5 decimal places.

(a) $(4 - x)^{-\frac{1}{2}} = \frac{1}{2}(1 - \frac{1}{4}x)^{-\frac{1}{2}}$.

$= \frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2 + ...$

(b) 4 − x with x = 0.04 gives 3.96.

Substitute: $\frac{1}{2} + \frac{1}{16}(0.04) + \frac{3}{256}(0.04)^2$.

= 0.50252 (5 d.p.).

Answer: (a) $\frac{1}{2} + \frac{1}{16}x + \frac{3}{256}x^2$ (b) 0.50252

4 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 72 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan \alpha = \frac{3}{4}$ and $\tan \beta = \frac{4}{3}$.

Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 81.6 N.

Horizontally: $T_1 \cos \alpha = T_2 \cos \beta$, so $T_1 \times \frac{4}{5} = T_2 \times \frac{3}{5}$

Vertically: $T_1 \sin \alpha + T_2 \sin \beta = 72$, and $T_2 = \frac{4}{3} T_1$

$T_1(\frac{3}{5} + \frac{4}{3} \times \frac{4}{5}) = 72$, so $T_1 = 43.2$ and $T_2 = 57.6$

The tension in BP is the larger: it is 57.6 when W = 72

Tensions are proportional to W, so $W_{max} = 72 \times \frac{81.6}{57.6} = 102$

Answer: (a) AP: 43.2 N, BP: 57.6 N (b) 102 N

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5 Segments: exact area and perimeter (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A chord AB of a circle, centre O and radius 4 cm, subtends an angle of $\frac{\pi}{2}$ at O.

Find the exact area and the exact perimeter of the minor segment cut off by AB.

Sector OAB: $\frac{1}{2}r^2\theta = \frac{1}{2} \times 4^2 \times \frac{\pi}{2}$; triangle OAB: $\frac{1}{2}r^2\sin\theta = 8$

Area = $4\pi - 8$

Chord AB = $2r\sin(\theta/2) = 4\sqrt{2}$

Arc AB = $r\theta = 2\pi$

Perimeter = arc + chord

Answer: Area = $-8 + 4\pi$ cm^2, perimeter = $2\pi + 4\sqrt{2}$ cm

6 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x^{\frac{2}{3}} + 4x^{\frac{1}{3}} + 3 = 0$.

Let $u = x^{\frac{1}{3}}$: $u^2 + 4u + 3 = 0$.

$(u + 1)(u + 3) = 0$, so $u = -1$ or $u = -3$.

Cube: $x = -1$ or $x = -27$.

Answer: $x = -27$ or $x = -1$

7 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of

$y = \frac{e^x}{x}$ for $x > 0$,

and determine its nature.

$dy/dx = \frac{e^x(x - 1)}{x^2} = 0$ gives $x = 1$

$y = e$ when $x = 1$

Second derivative at this point is positive, so it is a minimum

Answer: (1, e), minimum

8 Block on a plank: limiting slip (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A plank of mass 3 kg lies on a smooth horizontal table, with a block of mass 6 kg on it ($\mu = 0.5$ between them). A horizontal force P N acts on the block.

(a) Find the greatest P for which the block does not slip. (b) Find the accelerations of block and plank when P = 264.6. Take g = 9.8 m s^{-2}.

No slipping: common acceleration $a = \frac{P}{9}$. The plank is accelerated only by friction: $F = 3a$

Limiting friction: $F = \mu mg = 0.5 \times 6 \times 9.8 = 29.4$, so $a = \frac{29.4}{3} = 9.8$

$P_{max} = 9 \times 9.8 = 88.2$

$P = 264.6 > 88.2$, so the block slips. Plank: $a = \frac{29.4}{3} = 9.8$

Block: $264.6 - 29.4 = 6a$, so $a = 39.2$

Answer: (a) 88.2 N (b) block 39.2 m s^{-2}, plank 9.8 m s^{-2}

9 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two boats P and Q move so that, t hours after noon, their position vectors in km are

$\mathbf{r}_P = \begin{pmatrix}4\\5\end{pmatrix} + t\begin{pmatrix}0\\-2\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}2\\1\end{pmatrix} + t\begin{pmatrix}-1\\1\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-2\\-4\end{pmatrix} + t\begin{pmatrix}-1\\3\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}-1\\3\end{pmatrix} = 0$

$-10 + t(10) = 0$, so $t = 1$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-3\\-1\end{pmatrix}$, with length $\sqrt{10}$

Answer: t = 1, minimum distance = $\sqrt{10}$ km

10 Normal approximation to a binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(80, 0.4)$.

(a) State why a normal approximation is suitable for X.

(b) Use a normal approximation to estimate $P(28 \leq X \leq 40)$, to 3 significant figures.

np = 32 and nq = 48, both greater than 5 (n large, p not extreme)

$Y \sim N(32, 19.2)$ with $\sigma = 4.382$

Apply the continuity correction:

$P(28 \leq X \leq 40) \approx P(27.5 < Y < 40.5) = 0.9738 - 0.1522 = 0.822$

Answer: 0.822