Day 138 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Small angle approximations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (small angle))

θ is small and measured in radians. Use small angle approximations to find the value that $\frac{\sin 2\theta \tan 4\theta}{1 - \cos 4\theta}$ approaches as $\theta \to 0$.

$\sin 2\theta \approx 2\theta$, $\tan 4\theta \approx 4\theta$, $1 - \cos 4\theta \approx 8\theta^2$

Expression $\approx \frac{2\theta \times 4\theta}{8\theta^2} = 1$

Answer: 1

2 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 9 s, P is back at its starting point.

Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 3 \times 12 + \frac{1}{2}(12 - W)(3) - W(3) = 0$

$18 + 18 - 1.5W - 3W = 0$, so $4.5W = 36$

$W = 8$

v = 0 where the line from (3, 12) to (6, −8) crosses the axis: $t = 3 + 3 \times \frac{12}{20} = 4.8$

Greatest distance = area of triangle = $\frac{1}{2} \times 4.8 \times 12$

Answer: (a) W = 8 (b) 28.8 m

3 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A game costs £2. Two fair dice are rolled: a total of at least 10 wins £8, a total of at most 2 wins £2, otherwise nothing.

(a) Find the expected profit.

(b) Find the standard deviation of the profit (3 s.f.).

(c) Find the stake that makes the game fair.

$P(\text{total} \geq 10) = \frac{1}{6}$, $P(\text{total} \leq 2) = \frac{1}{36}$, otherwise $\frac{29}{36}$

Profit: £6 with $\frac{1}{6}$, £0 with $\frac{1}{36}$, £−2 with $\frac{29}{36}$

(a) $E = -\frac{11}{18}$

(b) $E(P^2) = \frac{83}{9}$, $\text{Var} = \frac{83}{9} - (-\frac{11}{18})^2 = \frac{2867}{324}$, so $\sigma = 2.975$

(c) Fair when the stake equals E(winnings) = $\frac{1}{6} \times 8 + \frac{1}{36} \times 2 = \frac{25}{18}$

Answer: (a) $-£\frac{11}{18}$ (b) 2.97 (c) $£\frac{25}{18}$

4 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(\mathbf{i} - 2\mathbf{j})$ N, $(3\mathbf{i} + k\mathbf{j})$ N and $(-7\mathbf{i} - 3\mathbf{j})$ N act on a particle of mass 2 kg. The resultant is parallel to $(-\mathbf{i})$.

Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(-3)\mathbf{i} + (-5 + k)\mathbf{j}$

Parallel to $(-\mathbf{i})$: $\frac{-5 + k}{-3} = \frac{0}{-1}$, so $k = 5$

$R = (-3\mathbf{i})$ N, so $|R| = \sqrt{(-3^2 + 0^2)} = 3.000$

$a = |R| \div 2$

Answer: (a) k = 5 (b) 1.50 m s^{-2}

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 15 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 90$, and after 10 minutes $\theta = 60$.
Find A and k, then find when $\theta = 31$ (3 s.f.).

$t = 0$: $15 + A = 90$ so $A = 75$

$t = 10$: $15 + 75e^{-10k} = 60$ so $e^{-10k} = \frac{3}{5}$

$k = \frac{1}{10} \ln \frac{5}{3} = 0.0511$

$75\, e^{-kt} = 16$ so $t = \frac{1}{k} \ln \frac{75}{16}$

$t = 30.2$ minutes

Answer: A = 75, k = 0.0511, t = 30.2 minutes

6 Stratified sample then probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A school has 200 students in Year 10, 80 in Year 11 and 200 in the sixth form.
A stratified sample of 36 students is taken.
(a) Find the number sampled from each group.
(b) Two students are chosen at random from the sample. Find the probability they are from the same group.

Sample sizes: $36 \times \frac{200}{480}$, $36 \times \frac{80}{480}$, $36 \times \frac{200}{480} = 15, 6, 15$

P(same) = 15 × 14 + 6 × 5 + 15 × 14/36 × 35

$= \frac{450}{1260} = 0.357$

Answer: (a) 15, 6, 15 (b) 0.357

7 Modulus inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $|x - 6| > -3x - 4$.

Boundary values: $x - 6 = -3x - 4$ gives $x = \frac{1}{2}$;

$x - 6 = -(-3x - 4)$ gives $x = -5$.
Also the corner of $|x - 6|$ is at $x = 6$.
Testing each region (check the graph of each side) gives the solution set.
Answer: $x > -5$

8 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = 3x^2 - 30x + 79$.

Complete the square: $y = 3(x - 5)^2 + 4$.
Step 1: $y = x^2 \rightarrow y = 3x^2$: a stretch, scale factor 3, parallel to the y-axis.
Step 2: replace x by $(x - 5)$ and add 4: translation by $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

Answer: a stretch, scale factor 3, parallel to the y-axis, then translation $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

9 Stationary points of trigonometric curves (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary points of
$y = 3x + 6\cos x$ for $0 \leq x \leq \pi$,
and determine their nature.

$dy/dx = 3 - 6\sin x = 0$, so $\sin x = \frac{1}{2}$

$x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$

$d^2y/dx^2 = -6\cos x$

Its sign at each point decides maximum or minimum

Answer: $(\frac{\pi}{6}, \frac{1}{2}\pi + 3\sqrt{3})$ maximum; $(\frac{5\pi}{6}, \frac{5}{2}\pi - 3\sqrt{3})$ minimum

10 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(27 + 2x)^{\frac{1}{3}}$ in ascending powers of x up to and including the term in x^3.
State the values of x for which it is valid.

$(27 + 2x)^{\frac{1}{3}} = 27^{\frac{1}{3}}(1 + \frac{2}{27}x)^{\frac{1}{3}} = 3(1 + u)^{\frac{1}{3}}$, $u = \frac{2}{27}x$.

$(1 + u)^{\frac{1}{3}} = 1 + \frac{1}{3}u + (-\frac{1}{9})u^2 + \frac{5}{81}u^3 + ...$

Substitute u and multiply by 3.

Valid when $|u| < 1$, i.e. $|x| < \frac{27}{2}$.

Answer: $3 + \frac{2}{27}x - \frac{4}{2187}x^2 + \frac{40}{531441}x^3 + ...$; $|x| < \frac{27}{2}$