Day 137 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Simultaneous logarithmic equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve the simultaneous equations
$\log_2 x + \log_2 y = 7$
$x - y = 28$

$\log_2(xy) = 7$, so $xy = 2^7 = 128$

$x = y + 28$: $y^2 + 28y - 128 = 0$
$(y - 4)(y + 32) = 0$; $y > 0$ so $y = 4$
Then $x = y + 28$
Answer: $x = 32$, $y = 4$

2 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{5}{(x-4)} < \frac{4}{(x+2)}$.

Multiply by the positive quantity $(x - 4)^2(x + 2)^2$:
$5(x - 4)(x - (-2))^2 < 4(x - (-2))(x - 4)^2$.
Move to one side and factorise: $(x - 4)(x - (-2))(x + 26) < 0$.
Critical values: 4, −2, −26. Test the four regions.
Answer: $x < -26$ or $-2 < x < 4$

3 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x e^y + 3y = 4$.
Find the equation of the tangent to C at the point (4, 0),
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At (4, 0): $dy/dx = -\frac{1}{7}$

$y - 0 = (-\frac{1}{7})(x - 4)$, then clear fractions

Answer: $x + 7y - 4 = 0$

4 Binomial expansion (fractional or negative powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the binomial expansion of $(4 - 2x)^{-\frac{1}{2}}$ in ascending powers of x up to and including the term in x^3.
State the values of x for which it is valid.

$(4 - 2x)^{-\frac{1}{2}} = 4^{-\frac{1}{2}}(1 - \frac{1}{2}x)^{-\frac{1}{2}} = \frac{1}{2}(1 + u)^{-\frac{1}{2}}$, $u = -\frac{1}{2}x$.

$(1 + u)^{-\frac{1}{2}} = 1 + (-\frac{1}{2})u + \frac{3}{8}u^2 + (-\frac{5}{16})u^3 + ...$

Substitute u and multiply by $\frac{1}{2}$.

Valid when $|u| < 1$, i.e. $|x| < 2$.

Answer: $\frac{1}{2} + \frac{1}{8}x + \frac{3}{64}x^2 + \frac{5}{256}x^3 + ...$; $|x| < 2$

5 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(11, p). It is given that $P(X = 2) = \frac{10}{3} \times P(X = 1)$.

(a) Find the value of p.
(b) Find P(X = 0), correct to 3 significant figures.

$\frac{110}{2} p^2(1 - p)^9 = \frac{10}{3} \times 11\, p(1 - p)^{10}$

Divide by $p(1 - p)^9$: $5p = \frac{10}{3}(1 - p)$

$(5 + \frac{10}{3})p = \frac{10}{3}$

$p = \frac{2}{5}$

$P(X = 0) = (1 - p)^{11} = (\frac{3}{5})^{11} = 0.00363$

Answer: (a) $p = \frac{2}{5}$ (b) 0.00363

6 Force up a rough slope: find weight and friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight W N is on a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{12}{13}$. A force P N acts up a line of greatest slope; the particle stays at rest for $34 \le P \le 62$.
Find (a) W, (b) the coefficient of friction, (c) the acceleration when P = 124. Take g = 9.8 m s^{-2}.

Limiting friction acts down the plane at P = 62 and up the plane at P = 34
$P_{max} = W \sin\alpha + F$ and $P_{min} = W \sin\alpha - F$, where $F = \mu W \cos\alpha$

Adding: $2W \sin\alpha = 96$, so $W = \frac{96}{2 \times \frac{12}{13}} = 52$

Subtracting: $2F = 28$, so $F = 14 = \mu \times 52 \times \frac{5}{13}$, giving $\mu = 0.7$

P = 124: net force up the plane = 124 − 62 = 62, and mass = $\frac{52}{9.8}$

Answer: (a) 52 N (b) 0.7 (c) 11.7 m s^{-2}

7 Cumulative distribution function (CIE)

The cdf of X is $F(x) = 0$ for $x < 0$, $F(x) = 1 - (1 - \frac{x}{4})^2$ for $0 \le x \le 4$, and $F(x) = 1$ for $x > 4$.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find P(1 < X < 2) and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{2}{4}(1 - \frac{x}{4})^1$ for $0 \le x \le 4$

(b) $E(X) = \int x\, f(x)\, dx$ from 0 to 4 = $\frac{4}{3}$

(c) $P = F(2) - F(1) = 0.3125$
Median m: $F(m) = 1/2$, so $m = 4(1 - (1/2)^{1/2})$
m = 1.172

Answer: (a) $f(x) = \frac{2}{4}(1 - \frac{x}{4})^1$ (b) $\frac{4}{3}$ (c) 0.312, median 1.17

8 Parametric to Cartesian (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations
$x = t^2$, $y = 2t^3$.
Show that $y^2 = kx^3$ and find k.

$y^2 = 4t^6$.
$x^3 = t^6$.
So $y^2 = 4x^3$.
Answer: k = 4

9 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3(\theta - \sin\theta)$, $y = 3(1 - \cos\theta)$.
Find the exact area under one arch, for $0 \le \theta \le 2\pi$,
between the curve and the x-axis.

$dx/d\theta = 3(1 - \cos\theta)$

Area $= \int_0^{2\pi} 3^2(1 - \cos\theta)^2 \, d\theta = 9\int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) \, d\theta$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$, so the integrand integrates to $2\pi + \frac{1}{2} \times 2\pi = 3\pi$

Answer: 27π

10 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (3 kg) lies on a rough table with $\mu = 0.2$, and is joined by a light string over a smooth pulley (see diagram) to particle B (3 kg) on a smooth slope, $\sin\alpha = \frac{3}{5}$.

Released from rest, the string is cut when B has moved 3.5 m. Find the further distance A slides before stopping.

A: $T - 0.2 \times 3 \times 9.8 = 3a$. B: $3 \times 9.8 \times \frac{3}{5} - T = 3a$

Adding: $9.8(3 \times \frac{3}{5} - 0.2 \times 3) = 6a$, so $a = 1.96$

$T = 3(1.96 + 0.2 \times 9.8) = 11.76$

When cut: $v^2 = 2 \times 1.96 \times 3.5 = 13.72$. Then A decelerates at $\mu g = 1.96$

Further distance $= \frac{13.72}{2 \times 1.96} = 3.5$

Answer: 3.5 m