Day 134 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 8 m. After each bounce it rises to $\frac{1}{2}$ of its previous height.

(a) Find the total distance it travels before it stops.
(b) After how many bounces does it rise less than 0.01 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$= H + \frac{2Hr}{1 - r} = 8 + \frac{2 \times 8 \times \frac{1}{2}}{1 - \frac{1}{2}} = 24.$

(b) Rebound height after n bounces $= 8(\frac{1}{2})^n < 0.01.$

$n > \frac{\log(0.00125)}{\log(0.5)} = 9.64$, so n = 10.

Answer: (a) 24 m (b) 10 bounces

2 Two-tailed binomial test (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A coin is said to land on blue with probability 0.5. In 40 trials it does so 29 times.
Test at the 10% level whether the probability is different from 0.5.

H_0: p = 0.5, H_1: p ≠ 0.5 (two-tailed). X ~ B(40, 0.5)
29 is above the mean 20, so use the upper tail: P(X ≥ 29) = 0.00321
Each tail is compared with 5%: 0.00321 is less than 5%
Reject H_0: there is sufficient evidence that the probability is different from 0.5
Answer: Reject H_0 (p-value 0.00321 against 5%)

3 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 8 strips to estimate

$\int_1^2 \sqrt{4 - x^2}\, dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

h = $\frac{1}{8}$; ordinates at x = 1, $\frac{9}{8}$, $\frac{5}{4}$, $\frac{11}{8}$, …, 2

Sum = $\frac{h}{2}[y_0 + y_8 + 2(y_1 + \ldots + y_7)] = 1.211$

f''(x) < 0 on the interval, so the curve is concave and the trapezia lie below the curve
Answer: 1.211; underestimate

4 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{3x - 2} = x - 2$.

Square both sides: $3x - 2 = (x - 2)^2$.
$x^2 - 7x + 6 = 0$, so x = 1 or x = 6.
Check in $\sqrt{3x - 2} = x - 2$: x = 6 gives 4 ≥ 0 (valid).
x = 1 gives right-hand side −1 < 0, so it is rejected.
Answer: x = 6

5 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from level ground with horizontal and vertical velocity components 19.6 and 24.5 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, |v_y| = v_x = 19.6
Moving upwards: 24.5 − 9.8t = 19.6, so t = 0.5
Moving downwards: 24.5 − 9.8t = −19.6, so t = 4.5
Horizontal distance = 19.6 × (4.5 − 0.5)
Answer: (a) 0.5 s and 4.5 s (b) 78.4 m

6 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 9 seats. Each row has 5 more seats than the row in front. Find
(a) the number of seats in row 59,
(b) the least number of rows for at least 1078 seats.

Arithmetic: $a = 9$, $d = 5$.
(a) $u_{59} = 9 + 58 \times 5 = 299$.

(b) $S_n = \frac{n}{2}(18 + 5n - 5) = \frac{n(5n + 13)}{2} \geq 1078$.

$S_{19} = 1026 < 1078$, $S_{20} = 1130 \geq 1078$.

Answer: (a) 299 (b) 20 rows

7 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (8 kg) and B (5 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.5$, and B is 1.3 m above the floor.
The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.5 \times 8 \times 9.8 = 8a$. B: $5 \times 9.8 - T = 5a$

Adding: $(5 - 0.5 \times 8) \times 9.8 = 13a$, so $a = \frac{49}{65}$

$v^2 = 2 \times \frac{49}{65} \times 1.3 = 1.96$, so $v = 1.4$

After B lands, A decelerates at $\mu g = 4.9$: extra distance $= \frac{1.4^2}{2 \times 4.9} = 0.2$

Total distance $= 1.3 + 0.2$

Answer: 1.5 m

8 R-form: greatest and least values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(\theta) = \frac{2}{13 + 8\sin\theta + 6\cos\theta}$

(a) Write $8\sin\theta + 6\cos\theta$ as $R\sin(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.
(b) Find the greatest and least values of $f(\theta)$, and the smallest positive θ at which the greatest value occurs, to 1 d.p.

$R = \sqrt{8^2 + 6^2} = 10$, $\tan\alpha = \frac{6}{8}$, $\alpha = 36.9°$

Denominator $= 13 + 10\sin(\theta + 36.9°)$, which lies between 3 and 23

Greatest $f(\theta) = \frac{2}{3} = \frac{2}{3}$, when $\sin(\theta + \alpha) = -1$

Least $f(\theta) = \frac{2}{23} = \frac{2}{23}$, when $\sin(\theta + \alpha) = 1$

$\sin(\theta + \alpha) = -1$: $\theta + 36.9° = 270°$, so $\theta = 233.1°$

Answer: (a) $10\sin(\theta + 36.9°)$ (b) greatest $\frac{2}{3}$, least $\frac{2}{23}$, $\theta = 233.1°$

9 Pdf of the form a + bx with a given mean (CIE)

The pdf of X is $f(x) = a + bx$ for $0 \leq x \leq 3$, and 0 otherwise, where a and b are constants. $E(X) = \frac{39}{20}$.

(a) Find a and b.
(b) Find $P(X < 1.5)$.

Total area = 1: $3a + 4.5b = 1$

$E(X) = \int x(a + bx)\,dx = 4.5a + 9b = \frac{39}{20}$

Solving the simultaneous equations: $a = \frac{1}{30}$, $b = \frac{1}{5}$

$P(X < 1.5) = \int f(x)\,dx$ from 0 to 1.5 $= \frac{1}{30} \times 1.5 + \frac{1}{5} \times 1.125$

$= \frac{11}{40}$

Answer: (a) $a = \frac{1}{30}$, $b = \frac{1}{5}$ (b) $\frac{11}{40}$

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix}5\\0\\5\end{pmatrix} + t\begin{pmatrix}3\\2\\0\end{pmatrix}$.

The point P has coordinates (0, 1, 6).

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix}-5\\1\\1\end{pmatrix} - t\begin{pmatrix}3\\2\\0\end{pmatrix}$

FP · d = 0: $-13 - t(13) = 0$, so $t = -1$

F = (2, −2, 5)

$\mathrm{PF} = \begin{pmatrix}-2\\3\\1\end{pmatrix}$, so $\mathrm{PF} = \sqrt{14}$

Answer: F = (2, −2, 5), PF = $\sqrt{14}$