Day 133 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Acceleration given, find when P returns to O (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t = 0, P passes through the fixed point O with velocity -96 m s^{-1}. At time t seconds, P has acceleration $(-6t + 44)$ m s^{-2}.
Find (a) the times at which P is at O again, (b) the greatest distance of P from O before it first returns to O.

$v = \int a\,dt = -3t^2 + 44t - 96$ ($v = -96$ when $t = 0$)
$s = \int v\,dt = -t^3 + 22t^2 - 96t$ ($s = 0$ when $t = 0$)
$s = -t(t - 6)(t - 16)$, so P is at O when $t = 6$ and $t = 16$

Greatest distance when $v = 0$: $-3t^2 + 44t - 96 = 0$, so $t = \frac{8}{3}$ (the smaller root)

$|s| = \frac{8}{3} \times \frac{10}{3} \times \frac{40}{3}$

Answer: (a) t = 6, t = 16 (b) $\frac{3200}{27}$ (= 119) m

2 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta - 3\cos\theta + 2 = 0$
for $0 \leq \theta < 360°$.

Use $\cos 2\theta \equiv 2\cos^2\theta - 1$: $2\cos^2\theta - 3\cos\theta + 1 = 0$
$(2\cos\theta - 1)(\cos\theta - 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 1$

Using the CAST diagram for $0 \leq \theta < 360°$
Answer: θ = 0°, 60°, 300°

3 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The fill volume of a bottle is N(μ, 36) with known variance 36. The mean was claimed to be 54 ml.
A random sample of 49 has mean 56.5. Test at the 5% level whether the mean is less than 54.

H_0: $\mu = 54$, H_1: $\mu < 54$. Under H_0, the sample mean ~ N(54, $\frac{36}{49}$)

$z = \frac{56.5 - 54}{6 \div \sqrt{49}} = 2.917$
p-value = P(Z < 2.917) = 0.998
0.998 > 5%, so do not reject H_0
There is insufficient evidence at the 5% level that the mean has changed as claimed
Answer: Do not reject H_0 (z = 2.917, p-value 0.998)

4 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of
$\sin(\sin^{-1}(\frac{24}{25}) + \cos^{-1}(\frac{21}{29}))$.

Let $A = \sin^{-1}(\frac{24}{25})$ and $B = \cos^{-1}(\frac{21}{29})$, both acute

$\sin A = \frac{24}{25}$, $\cos A = \frac{7}{25}$, $\cos B = \frac{21}{29}$, $\sin B = \frac{20}{29}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{24}{25} \times \frac{21}{29} + \frac{7}{25} \times \frac{20}{29}$

Answer: $\frac{644}{725}$

5 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\sqrt{17}$ is irrational.

Assume $\sqrt{17} = \frac{a}{b}$ with a, b integers and no common factor.

Then $17\,b^2 = a^2$, so a^2 is a multiple of 17.
As 17 is prime, a is a multiple of 17: let $a = 17k$.
Then $17\,b^2 = 289\,k^2$, so $b^2 = 17\,k^2$, and b is a multiple of 17.
a and b have common factor 17: a contradiction. Hence $\sqrt{17}$ is irrational.
Answer: Proved

6 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(12, p). It is given that $P(X = 2) = \frac{11}{3} \times P(X = 1)$.

(a) Find the value of p.
(b) Find P(X = 0), correct to 3 significant figures.

$\frac{132}{2}\,p^2(1-p)^{10} = \frac{11}{3} \times 12\,p(1-p)^{11}$

Divide by $p(1-p)^{10}$: $5.5\,p = \frac{11}{3}(1-p)$

$(5.5 + \frac{11}{3})p = \frac{11}{3}$

$p = \frac{2}{5}$

$P(X = 0) = (1-p)^{12} = (\frac{3}{5})^{12} = 0.00218$

Answer: (a) $p = \frac{2}{5}$ (b) 0.00218

7 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 8 strips to estimate

$\int_1^3 \frac{1}{1+x^2}\,dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{4}$; ordinates at $x = 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, \ldots, 3$

$\text{Sum} = \frac{h}{2}[y_0 + y_8 + 2(y_1 + \ldots + y_7)] = 0.4659$

f''(x) > 0 on the interval, so the curve is convex and the trapezia lie above the curve
Answer: 0.4659; overestimate

8 Sigma notation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sum_{r=1}^{n}(5r-1) = 1030$, find n.

$\sum_{r=1}^{n}(5r-1) = 5 \times \frac{n(n+1)}{2} - n.$

$\frac{5\,n(n+1)}{2} - n = 1030$, so $5n^2 + 3n - 2060 = 0$.

Solving (n > 0): n = 20.
Answer: n = 20

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9 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 22 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 22 m horizontally and 17 m vertically from O.

Find the two possible values of $\tan\theta$, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 22\cos\theta \times t$. Vertical: $y = 22\sin\theta \times t - 4.9t^2$

Eliminating t: $y = x\tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 22^2}{2 \times 22^2} = 4.9$, so $17 = 22\tan\theta - 4.9(1 + \tan^2\theta)$

$4.9\tan^2\theta - 22\tan\theta + 21.9 = 0$

$\tan\theta = \frac{73}{49}$ or 3, so $\theta = 56.1°$ or $71.6°$

Answer: $\tan\theta = \frac{73}{49}$ or 3; $\theta = 56.1°$ or $71.6°$

10 Sums of multiples (arithmetic series) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the sum of all the integers from 1 to 773 inclusive that are divisible by 2 or by 9, but not by both.

Multiples of 2: 386 terms (2 to 772): sum = 149382.
Multiples of 9: 85 terms: sum = 32895.
Multiples of both, i.e. of 18: 42 terms: sum = 16254.
Required sum = 149382 + 32895 − 2 × 16254.
Answer: 149769