Day 132 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 1$.
Show that $(x - 1)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(1) = 1 - 1 = 0$, so $(x - 1)$ is a factor.
$f(x) = (x - 1)(x^2 + x + 1)$.
For $x^2 + x + 1 = 0$: $b^2 - 4ac = 1 - 4 = -3 < 0$.
No real roots from the quadratic, so the only real root is $x = 1$.
Answer: Proved; the only real root is $x = 1$

2 Independence from a Venn diagram (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The Venn diagram shows probabilities; A and B are independent.
Find the possible values of x.

$P(A) = 0.15 + x$, $P(B) = 0.35 + x$, $P(A \cap B) = x$
Independent: $x = (0.15 + x)(0.35 + x)$
$x^2 - 0.5x + 0.0525 = 0$
$(x - 0.15)(x - 0.35) = 0$
Both values keep every probability between 0 and 1
Answer: $x = 0.15$ or $x = 0.35$

3 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $8\sec^2\theta + 14\tan\theta - 12 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $8\tan^2\theta + 14\tan\theta - 4 = 0$
$2(\tan\theta + 2)(4\tan\theta - 1) = 0$
$\tan\theta = -2$ or $\tan\theta = \frac{1}{4}$
$\theta = 116.6°$ and $14.0°$ $(+180°)$
Answer: $\theta = 14.0°, 116.6°, 194.0°, 296.6°$

4 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coefficient of x^3 in the expansion of
$(2x + 1)(1 - 3x)^7$.

$(1 - 3x)^7$: x^3 term has coefficient $C(7,3) \times (-3)^3 = -945$, x^2 term has coefficient $C(7,2) \times (-3)^2 = 189$.
x^3 in the product: $1 \times (-945) + 2 \times 189$.
$= -567$.
Answer: -567

5 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from level ground with horizontal and vertical velocity components 24.5 and 29.4 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take $g = 9.8$ m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 24.5$
Moving upwards: $29.4 - 9.8t = 24.5$, so $t = 0.5$
Moving downwards: $29.4 - 9.8t = -24.5$, so $t = 5.5$
Horizontal distance $= 24.5 \times (5.5 - 0.5)$
Answer: (a) 0.5 s and 5.5 s (b) 122.5 m

6 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 + n + 5$ is prime for every positive integer n.
Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.
For n = 4: $4^2 + 4 + 5 = 25$.
$25 = 5 \times 5$, which is not prime.
All smaller values of n give a prime, so n = 4 is the smallest.
Answer: n = 4 (value 25 = 5 × 5)

7 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curves $y = e^x$ and $y = 36\,e^{-x}$ meet at the point P.
Find the exact area of the region bounded by the two curves and the y-axis.

$e^x = 36\,e^{-x}$ gives $e^{2x} = 36$, so $e^x = 6$, $x = \ln 6$

Area $= \int_0^{\ln 6} (36\,e^{-x} - e^x)\,dx = [-36\,e^{-x} - e^x]$

$= (-6 - 6) - (-36 - 1)$
Answer: 25

8 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^x \cos 2x$. Show that $dy/dx = e^x(\cos 2x - 2 \sin 2x)$.
Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (1)e^x \cos 2x - e^x \times 2 \sin 2x$
$dy/dx = 0$ requires $1 \cos 2x = 2 \sin 2x$
$\tan 2x = \frac{1}{2}$, smallest positive $2x = 0.4636$
$x = 0.232$
Second derivative test (or gradient change): a maximum
Answer: x = 0.232, maximum

9 Normal approximation to a Poisson (CIE)

The number of customers arriving at a shop in one hour is Po(16).
Use a suitable approximation to find $P(X \geq 24)$, to 3 significant figures.

$\lambda = 16$ is large (greater than 15), so X is approximately N(16, 16)
Standard deviation $= \sqrt{16} = 4.000$
Continuity correction:
$P(X \geq 24) \approx P(Y > 23.5) = P(Z > 1.875) = 0.0304$
Answer: 0.0304

10 Just clearing a wall at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 45° to the horizontal. The ball just passes over the top of a vertical wall of height 34.8 m, at a horizontal distance of 74 m from O.
Find (a) the speed of projection, (b) the greatest height of the ball above the ground, to 3 s.f. Take $g = 9.8\ m\ s^{-2}$.

At 45°: $x = \frac{u\,t}{\sqrt{2}}$ and $y = \frac{u\,t}{\sqrt{2}} - 4.9t^2$, so $y = x - \frac{9.8x^2}{u^2}$

$34.8 = 74 - \frac{9.8 \times 74^2}{u^2}$, so $u^2 = \frac{9.8 \times 5476}{39.2} = 1369$

$u = 37$

Greatest height $= \frac{(u \sin 45°)^2}{(2 \times 9.8)} = \frac{1369}{39.2}$

Answer: (a) 37 m s^{-1} (b) 34.9 m