Day 131 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sin(2\theta + 10°) = -\frac{\sqrt{3}}{2}$
for $0° \leq \theta \leq 180°$.

Let $u = 2\theta + 10°$; then $\sin u = -\frac{\sqrt{3}}{2}$
Base solutions: $u = 240, -60°$ (+ multiples of 360°)
$\theta = (u - 10°) \div 2$, keeping those in range
Answer: θ = 115°, 145°

2 Poisson: scaling the rate, then binomial (CIE)

Calls arrive at a switchboard at an average rate of 6 per hour, independently and at random.
(a) Find the probability of at least 2 in a 30-minute period.
(b) Find the probability that, of 6 separate 30-minute periods, exactly 2 have no arrivals.

30-minute period: $\lambda = 6 \times \frac{30}{60} = 3$, so $X \sim Po(3)$
(a) $P(X \geq 2) = 1 - P(X \leq 1) = 0.8009$
(b) $P(X = 0) = e^{-3} = 0.04979$
$Y \sim B(6, 0.04979)$: $P(Y = 2) = 0.0303$
Answer: (a) 0.801 (b) 0.0303

3 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 + 3n + 19$ is prime for every positive integer n.
Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.
For n = 15: $15^2 + 315 + 19 = 289$.
$289 = 17 \times 17$, which is not prime.
All smaller values of n give a prime, so n = 15 is the smallest.
Answer: n = 15 (value 289 = 17 × 17)

4 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(-4\mathbf{i} - 6\mathbf{j})$ N, $(7\mathbf{i} + k\mathbf{j})$ N and $(2\mathbf{i} + \mathbf{j})$ N act on a particle of mass 4 kg. The resultant is parallel to $(5\mathbf{i} - 11\mathbf{j})$.
Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant = $(5)\mathbf{i} + (-5 + k)\mathbf{j}$
Parallel to $(5\mathbf{i} - 11\mathbf{j})$: $\frac{-5 + k}{5} = \frac{-11}{5}$, so $k = -6$
$\mathbf{R} = (5\mathbf{i} - 11\mathbf{j})$ N, so $|R| = \sqrt{(5^2 + -11^2)} = 12.08$
$a = |R| \div 4$
Answer: (a) k = −6 (b) 3.02 m s^{-2}

5 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{\sin 2\theta}{1 + \cos 2\theta} \equiv \tan\theta$.
(b) Hence solve $\frac{\sin 2\theta}{1 + \cos 2\theta} = -\frac{1}{2}$ for $0 \leq \theta < 360°$ (1 d.p.).

$\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2 \sin\theta \cos\theta}{2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$\tan\theta = -\frac{1}{2}$, so $\theta = -26.6°$ (principal value) and add 180°
Answer: θ = 153.4°, 333.4°

6 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Each year Jo pays £600 into an account on 1 January.
Interest of 2% is added each 31 December.
Find the first year-end at which the balance exceeds £8,000.

After n years: $600(1.02) + 600(1.02^2) + \ldots + 600(1.02^n)$.

GP: $a = 600 \times 1.02$, $r = 1.02$: $S_n = \dfrac{600 \times 1.02\,(1.02^n - 1)}{0.02}$.

$S_n > 8000 \Rightarrow 1.02^n > 1.2614 \Rightarrow n > 11.728$.

So n = 12 years.

Answer: after 12 years

7 Car on a hill: power and maximum speed (CIE)

A car of mass 800 kg, with engine power 34.6 kW, climbs a straight road with $\sin\alpha = \frac{1}{10}$ against a constant resistance of 600 N.
Find (a) the greatest steady speed up the hill, (b) the acceleration at 6.25 m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force $= 600 + 800 \times 9.8 \times \frac{1}{10} = 1384$ N

$P = F v$, so $v = \dfrac{34600}{1384} = 25$

At 6.25 m s^{-1}: $F = \dfrac{34600}{6.25} = 5536$ N

$F - 1384 = 800\,a$, so $a = \dfrac{4152}{800}$

Answer: (a) 25 m s^{-1} (b) 5.19 m s^{-2}

8 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 2\sin t$, $y = 2\cos t$,
for $0 \le t \le \frac{\pi}{2}$. Find the exact area bounded by the curve
and the coordinate axes.

$dx/dt = 2\cos t$

Area $= \int_0^{\pi/2} 2\cos t \times 2\cos t\,dt = 4\int_0^{\pi/2} \cos^2 t\,dt$

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$: integral $= 4 \times \frac{\pi}{4}$

Answer: 1π

9 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased coin lands on 'win' with the same unknown probability p each time. It is used 5 times in round 1 and 7 times in round 2.
Given that exactly 8 wins occur altogether, find the probability that exactly 5 of them were in round 1.

Let $X \sim B(5, p)$, $Y \sim B(7, p)$ be independent; $X + Y \sim B(12, p)$

$P(X = 5 \mid X + Y = 8) = \dfrac{P(X = 5)P(Y = 3)}{P(X + Y = 8)}$

$= \dfrac{C(5,5)C(7,3)\,p^8(1-p)^4}{C(12,8)\,p^8(1-p)^4}$

The powers of p and $(1 - p)$ cancel, leaving $\dfrac{1 \times 35}{495}$

Answer: $\frac{7}{99} = 0.0707$

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10 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations

$x = 3t$, $y = 4t^2$. The line $y = \frac{4}{3}x + 8$

meets C at A and B. Find the exact length AB.

Substitute: $4t^2 = 4t + 8$.

$4t^2 - 4t - 8 = 0 \Rightarrow t = 2$ or $t = -1$.

A(6, 16), B(−3, 4).

$AB^2 = 9^2 + 12^2 = 225$.

Answer: AB = 15