**Day 130** **A Level Maths: S Tier 10-a-day: Model Solutions**

## 1 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.

P(1, 2) lies on the circle: $(1 - r)^2 + (2 - r)^2 = r^2$.

$r^2 - 6r + 5 = 0$, so r = 1 or r = 5.

The diagram shows P on the larger circle, so r = 5.

$(x - 5)^2 + (y - 5)^2 = 25$, then expand.

**Answer: $x^2 + y^2 - 10x - 10y + 25 = 0$**

## 2 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A tennis ball is projected from a point O on horizontal ground with speed 49 m $s^{-1}$ at an angle α above the horizontal, where $\tan \alpha = \frac{3}{4}$. Find (a) the greatest height, (b) the speed when it is at half of this height (3 s.f.). Take g = 9.8 m $s^{-2}$.

Vertical: $0 = 29.4^2 - 2 \times 9.8 \times H$, so H = 44.1

At height $\frac{H}{2} = 22.05$: $v_y^2 = 29.4^2 - 2 \times 9.8 \times 22.05 = 432.18$

$v_x = 39.2$ (constant), so $\text{speed}^2 = 1536.64 + 432.18 = 1968.82$

**Answer: (a) 44.1 m (b) 44.4 m $s^{-1}$**

## 3 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx - 9}{x - 4}$ for $x \neq 4$. Given that f is self-inverse, find k, and hence solve f(x) = x.

Let y = f(x): $y(x - 4) = kx - 9$, so $x(y - k) = 4y - 9$.

$f^{-1}(x) = \frac{4x - 9}{x - k}$.

$f = f^{-1}$ requires k = 4.

f(x) = x: $4x - 9 = x(x - 4)$, so $x^2 - 8x + 9 = 0$.

$x = 4 \pm \sqrt{7} = 4 \pm \sqrt{7}$.

**Answer: k = 4; $x = 4 \pm \sqrt{7}$**

## 4 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 1 m $s^{-2}$ to a speed of V m $s^{-1}$, travels at this speed, then decelerates uniformly at 2 m $s^{-2}$ to rest.

The journey takes 67 s and covers 1040 m. Find V and the time spent at constant speed.

Time accelerating = V, time decelerating = $\frac{V}{2}$, so time at constant speed = $67 - V - \frac{V}{2}$

Area under the speed-time graph: $1040 = 67V - \frac{V^2}{2} - \frac{V^2}{4}$

Multiply by 2ab: $3V^2 - 268V + 4160 = 0$

V = 20 or $V = \frac{208}{3}$

$V = \frac{208}{3}$ gives time at constant speed −37 s, which is negative, so V = 20

Time at constant speed = 67 − 20 − 10 = 37 s

**Answer: V = 20, 37 s**

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## 5 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2\sec^2\theta + 9\tan\theta + 7 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $2\tan^2\theta + 9\tan\theta + 9 = 0$
$(2\tan\theta + 3)(\tan\theta + 3) = 0$
$\tan\theta = -\frac{3}{2}$ or $\tan\theta = -3$
$\theta = 123.7°$ and $108.4°$ $(+180°)$
**Answer: $\theta$ = 108.4°, 123.7°, 288.4°, 303.7°**

## 6 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The size of a rod is $N(120, 3^2)$ mm. A rod is 'large' if it exceeds 122.
(a) Find the probability that a rod is large.
(b) 18 rods are chosen at random. Find the probability that at least 2 are large.

(a) $P(X > 122) = P(Z > 0.667) = 0.2525$
(b) Let Y = number large, $Y \sim B(18, 0.2525)$
$P(Y \geq 2) = 1 - P(Y \leq 1) = 0.962$
**Answer: (a) 0.252 (b) 0.962**

## 7 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^1 xe^{-x}\,dx$

Let $u = x$ and $dv/dx = e^{-x}$
Integral = $[uv] - \int v\,du/dx\,dx$
Evaluate between the limits
**Answer: $1 - 2e^{-1}$**

## 8 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle with equation $(x + 3)^2 + (y + 2)^2 = 169$
passes through P(−15, −7). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre (−3, −2). Gradient of radius CP = $\frac{5}{12}$.
Tangent gradient = $-\frac{12}{5}$: $-12x - 5y = 215$.
$x = 0$: $y = -43$. $y = 0$: $x = -\frac{215}{12}$.
Area = ½ × $\frac{215}{12}$ × 43.
**Answer: $\frac{9245}{24}$**

## 9 Conditional probability: independent binomial counts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A biased coin lands on 'win' with the same unknown probability p each time. It is used 5 times in round 1 and 9 times in round 2.
Given that exactly 6 wins occur altogether, find the probability that exactly 3 of them were in round 1.

Let $X \sim B(5, p)$, $Y \sim B(9, p)$ be independent; $X + Y \sim B(14, p)$
$P(X = 3 \mid X + Y = 6) = \frac{P(X = 3)P(Y = 3)}{P(X + Y = 6)}$
$= \frac{C(5,3)C(9,3)\,p^6(1-p)^8}{C(14,6)\,p^6(1-p)^8}$
The powers of p and $(1 - p)$ cancel, leaving $\frac{10 \times 84}{3003}$
**Answer: $\frac{40}{143}$ = 0.280**

## 10 Closest approach of moving points (Edexcel, OCR, MEI, Eduqas, CIE (vectors in kinematics))

Two boats P and Q move so that, t hours after noon, their position vectors in km are

$\mathbf{r}_P = \begin{pmatrix}-5\\-5\end{pmatrix} + t\begin{pmatrix}-3\\2\end{pmatrix}$ and $\mathbf{r}_Q = \begin{pmatrix}-11\\-5\end{pmatrix} + t\begin{pmatrix}0\\-1\end{pmatrix}$.

Find the time at which they are closest and the exact minimum distance between them.

$\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-6\\0\end{pmatrix} + t\begin{pmatrix}3\\-3\end{pmatrix}$

Distance$^2 = |\mathbf{r}_Q - \mathbf{r}_P|^2$, minimised when $(\mathbf{r}_Q - \mathbf{r}_P) \cdot \begin{pmatrix}3\\-3\end{pmatrix} = 0$

$-18 + t(18) = 0$, so $t = 1$

Then $\mathbf{r}_Q - \mathbf{r}_P = \begin{pmatrix}-3\\-3\end{pmatrix}$, with length $\sqrt{18}$

**Answer: t = 1, minimum distance = $3\sqrt{2}$ km**