Day 13 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Double angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\cos 2\theta - 7\cos\theta + 4 = 0$
for $0 \leq \theta < 360°$.

Use $\cos 2\theta \equiv 2\cos^2\theta - 1$: $2\cos^2\theta - 7\cos\theta + 3 = 0$
$(2\cos\theta - 1)(\cos\theta - 3) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 3$ (impossible, $|\cos\theta| \leq 1$)

Using the CAST diagram for $0 \leq \theta < 360°$

Answer: $\theta = 60°, 300°$

2 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle $x^2 + y^2 + 2x - 6y - 159 = 0$
passes through P(−13, 8). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre (−1, 3). Gradient of radius CP = $-\frac{5}{12}$.

Tangent gradient = $\frac{12}{5}$: $-12x + 5y = 196$.

$x = 0$: $y = \frac{196}{5}$. $y = 0$: $x = -\frac{49}{3}$.

Area = $\frac{1}{2} \times \frac{49}{3} \times \frac{196}{5}$.

Answer: $\frac{4802}{15}$

3 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level with speed 58.8 m s^{-1}.
Find the length of time for which the ball is more than 171.5 m above the ground, and the greatest height it reaches. Take $g = 9.8$ m s^{-2}.

At height 171.5 m: $v^2 = 58.8^2 - 2 \times 9.8 \times 171.5 = 96.04$, so $v = 9.8$

Time from this height to the top: $t = \frac{9.8}{9.8} = 1$ s

By symmetry the time above this height is $2 \times 1 = 2$ s

Greatest height = $\frac{58.8^2}{(2 \times 9.8)} = 176.4$

Answer: 2 s, 176.4 m

4 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx + 2}{x - 4}$ for $x \neq 4$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 4) = kx + 2$, so $x(y - k) = 4y + 2$.

$f^{-1}(x) = \frac{4x + 2}{x - k}$.

$f = f^{-1}$ requires $k = 4$.

$f(x) = x$: $4x + 2 = x(x - 4)$, so $x^2 - 8x - 2 = 0$.

$x = 4 \pm \sqrt{18} = 4 \pm 3\sqrt{2}$.

Answer: $k = 4$; $x = 4 \pm 3\sqrt{2}$

5 Dropped and projected particles meet (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Ball A is projected vertically upwards from ground level with speed 11 m s^{-1}. At the same instant, ball B is released from rest 11 m directly above A.
The balls collide. Find (a) the height above the ground at which they collide, (b) the velocity of A just before the collision.
Take g = 9.8 m s^{-2}.

Relative to B, A moves upwards at the constant speed 11 m s^{-1} (both have acceleration g)
So the balls meet when 11 t = 11, giving t = 1
Height of A: $s = 11 \times 1 - \frac{1}{2} \times 9.8 \times 1^2 = 6.1$
$v = u - gt = 11 - 9.8 \times 1 = 1.2$
Answer: (a) 6.1 m (b) 1.2 m s^{-1} upwards

6 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = y^3 - y^2 - 3y$.
Find dy/dx and d^2y/dx^2 when $y = -1$.

$dx/dy = 3y^2 - 2y - 3 = 2$ when $y = -1$, so $dy/dx = \frac{1}{dx/dy} = \frac{1}{2}$

$d^2x/dy^2 = -8$ when $y = -1$

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{-8}{2^3}$

Answer: $dy/dx = \frac{1}{2}$, $d^2y/dx^2 = 1$

7 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A game costs £2. Two fair dice are rolled: a total of at least 11 wins £8, a total of at most 2 wins £2, otherwise nothing.
(a) Find the expected profit.
(b) Find the standard deviation of the profit (3 s.f.).
(c) Find the stake that makes the game fair.

P(total ≥ 11) = $\frac{1}{12}$, P(total ≤ 2) = $\frac{1}{36}$, otherwise $\frac{8}{9}$

Profit: £6 with $\frac{1}{12}$, £0 with $\frac{1}{36}$, £−2 with $\frac{8}{9}$

(a) $E = -\frac{23}{18}$

(b) $E(P^2) = \frac{59}{9}$, $Var = \frac{59}{9} - (-\frac{23}{18})^2 = \frac{1595}{324}$, so $\sigma = 2.219$

(c) Fair when the stake equals E(winnings) = $\frac{1}{12} \times 8 + \frac{1}{36} \times 2 = \frac{13}{18}$

Answer: (a) $-£\frac{23}{18}$ (b) 2.22 (c) $£\frac{13}{18}$

8 Remainder theorem (quadratic divisor) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

When a polynomial f(x) is divided by (x − 2)
the remainder is −6; when divided by (x + 1)
the remainder is 6. Find the remainder when
f(x) is divided by (x − 2)(x + 1).

$f(x) = (x - 2)(x + 1)Q(x) + px + q$.
$f(2) = -6$: $2p + q = -6$.
$f(-1) = 6$: $-p + q = 6$.
Subtracting: $3p = -12$, so $p = -4$, $q = 2$.
Answer: $-4x + 2$

9 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is obtuse. $\sin A = \frac{4}{5}$ and $\sin B = \frac{7}{25}$.

Find the exact values of cos(A + B) and cos(A − B).

$\cos A = \frac{3}{5}$, $\sin A = \frac{4}{5}$; $\sin B = \frac{7}{25}$, $\cos B = -\frac{24}{25}$

$\cos(A + B) = \cos A \cos B - \sin A \sin B = -\frac{4}{5}$

$\cos(A - B) = \cos A \cos B + \sin A \sin B = -\frac{44}{125}$

Answer: $\cos(A + B) = -\frac{4}{5}$, $\cos(A - B) = -\frac{44}{125}$

10 Two-tailed binomial test (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A coin is said to show heads with probability 0.4. In 30 trials it does so 4 times.

Test at the 10% level whether the probability is different from 0.4.

H_0: p = 0.4, H_1: p ≠ 0.4 (two-tailed). X ~ B(30, 0.4)

4 is below the mean 12, so use the lower tail: P(X ≤ 4) = 0.00151

Each tail is compared with 5%: 0.00151 is less than 5%

Reject H_0: there is sufficient evidence that the probability is different from 0.4

Answer: Reject H_0 (p-value 0.00151 against 5%)