Day 129 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Two particles over a pulley: height reached (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles of mass 7 kg and 3 kg hang on a light string over a smooth fixed pulley. The 7 kg particle is 2.75 m above the ground; the 3 kg one is on the ground.

Released from rest, find (a) the tension, (b) the greatest height reached by the 3 kg particle. Take $g = 9.8$ m s^{-2}.

$7g - T = 7a$ and $T - 3g = 3a$, so $a = \frac{7-3}{7+3} \times 9.8 = 3.92$

$T = 7(9.8 - 3.92) = 41.16$

The 3 kg particle rises 2.75 m before the 7 kg particle lands, reaching $v^2 = 2 \times 3.92 \times 2.75 = 21.56$

The string goes slack, so it then moves freely under gravity: extra height $= \frac{v^2}{2g} = \frac{21.56}{19.6} = 1.1$

Greatest height $= 2.75 + 1.1$

Answer: (a) 41.16 N (b) 3.85 m

2 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 8 components, each faulty with probability 0.15, independently. A box is rejected if it has at least 3 faulty components.

(a) Find the probability that a box is rejected.

(b) 10 boxes are inspected. Find the probability that at least 2 are rejected.

Let $X \sim B(8, 0.15)$. $P(\text{rejected}) = P(X \geq 3) = 1 - P(X \leq 2) = 0.1052$

Let Y = number of rejected boxes, $Y \sim B(10, 0.1052)$

$P(Y \geq 2) = 1 - P(Y \leq 1) = 0.284$

Answer: (a) 0.105 (b) 0.284

3 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + n + 3$, $u_1 = 2$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 1499$.

$u_n = u_1 + \sum_{k=1}^{n-1}(k + 3) = 2 + 1 \times \frac{(n-1)n}{2} + 3(n-1)$.

$= \frac{1}{2}n^2 + \frac{5}{2}n - 1$.

So $a = \frac{1}{2}$, $b = \frac{5}{2}$, $c = -1$.

$u_{52} = 1481 \leq 1499$ and $u_{53} = 1536 > 1499$.

Answer: (a) $a = \frac{1}{2}$, $b = \frac{5}{2}$, $c = -1$ (b) $n = 53$

4 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 9 and variance 3.6.

(a) Find n and p.

(b) Find P(X = 9), correct to 3 significant figures.

$np = 9$ and $np(1 - p) = 3.6$

Divide: $1 - p = 3.6 \div 9 = \frac{2}{5}$, so $p = \frac{3}{5}$

$n = 9 \div \frac{3}{5} = 15$

$P(X = 9) = C(15, 9)\left(\frac{3}{5}\right)^9\left(\frac{2}{5}\right)^6 = 0.207$

Answer: (a) $n = 15$, $p = \frac{3}{5}$ (b) 0.207

5 Velocity with a square-root term (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. At time t seconds, where $t \geq 0$, its velocity is v m s^{-1}, where $v = 40\sqrt{t} - 8t$.

Find (a) the greatest speed of P, (b) the greatest displacement of P from its starting point.

$\frac{dv}{dt} = \frac{40}{2\sqrt{t}} - 8 = 0$ when $\sqrt{t} = \frac{5}{2}$, so $t = \frac{25}{4}$

$v = 40 \times \frac{5}{2} - 8 \times \frac{25}{4} = 50$

$v = 0$ when $\sqrt{t}(40 - 8\sqrt{t}) = 0$: $t = 25$ ($t = 0$ is the start)

$s = \int v\,dt = \frac{2}{3} \times 40\, t^{\frac{3}{2}} - \frac{8}{2}t^2$, so at $t = 25$: $s = \frac{10000}{3} - 2500$

Answer: (a) 50 m s^{-1} (b) $\frac{2500}{3}$ m

6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 2\cos t$, $y = 4\sin t$.

Find a cartesian equation of the curve, with no fractions.

$\cos t = \frac{x}{2}$, $\sin t = \frac{y}{4}$

Use $\cos^2 t + \sin^2 t = 1$: $\frac{x^2}{4} + \frac{y^2}{16} = 1$

Multiply by 64

Answer: $16x^2 + 4y^2 = 64$

7 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{4x + 12} = 2x + 2$.

Square both sides: $4x + 12 = (2x + 2)^2$.

$4x^2 + 4x - 8 = 0$, so $x = -2$ or $x = 1$.

Check in $\sqrt{4x + 12} = 2x + 2$: $x = 1$ gives $4 \geq 0$ (valid).

$x = -2$ gives right-hand side $-2 < 0$, so it is rejected.

Answer: $x = 1$

8 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coordinates of the points of inflection of

$y = \frac{3}{x^2 + 27}$.

$dy/dx = -\frac{6x}{(x^2 + 27)^2}$

$d^2y/dx^2 = \frac{6(3x^2 - 27)}{(x^2 + 27)^3}$

Zero when $x^2 = 9$, so $x = \pm 3$, and the sign changes either side

$y = \frac{3}{36} = \frac{1}{12}$

Answer: $(3, \frac{1}{12})$ and $(-3, \frac{1}{12})$

9 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

In the expansion of $(1 + kx)^n$, where n is a positive integer, the coefficient of x is -15 and the coefficient of x^2 is 90. Find n and k.

Coefficient of x: $nk = -15$. Coefficient of x^2: $\frac{n(n-1)}{2}k^2 = 90$.

Divide the second by $(nk)^2$: $\frac{n-1}{2n} = \frac{90}{225}$.

$2n \times 90 = 225(n-1) \Rightarrow n = \frac{225}{45} = 5$.

Then $k = \frac{-15}{5} = -3$.

Answer: $n = 5$, $k = -3$

10 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 4 \sin t$, $y = 4 \cos t$,

for $0 \leq t \leq \frac{\pi}{2}$. Find the exact area bounded by the curve

and the coordinate axes.

$dx/dt = 4 \cos t$

Area $= \int_0^{\pi/2} 4 \cos t \times 4 \cos t \, dt = 16 \int_0^{\pi/2} \cos^2 t \, dt$

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$: integral $= 16 \times \frac{\pi}{4}$

Answer: 4π