Day 128 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Coded regression line (AQA, Edexcel, OCR, MEI)

Data pairs (x, y) are coded by $u = x - 8$ and $v = \frac{y}{4}$. The regression line of v on u is $v = 4.2 + 0.5u$.

(a) Find the regression line of y on x in the form $y = a + bx$.
(b) Use it to estimate y when $x = 12$.

Substitute $u = x - 8$ and $v = y/4$: $\frac{y}{4} = 4.2 + 0.5(x - 8)$

$y = 4[4.2 + 0.5x - 4] = 4 \times 0.2 + 4 \times 0.5x$
$y = 0.8 + 2x$
At $x = 12$: $y = 0.8 + 2 \times 12 = 24.8$
Answer: (a) $y = 0.8 + 2x$ (b) 24.8

2 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-2 + 4\sqrt{5}}{4 - \sqrt{5}}$ in the form $p + q\sqrt{5}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(4 + \sqrt{5})$.
Numerator: $(-2 + 4\sqrt{5})(4 + \sqrt{5}) = 12 + 14\sqrt{5}$.
Denominator: $4^2 - 1^2(5) = 11$.
So the fraction = $\frac{12 + 14\sqrt{5}}{11}$.
Answer: $\frac{12}{11} + \frac{14}{11}\sqrt{5}$

3 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 32 cm and its area is 55 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 32$ and $\frac{1}{2}r^2\theta = 55$

$r\theta = 32 - 2r$, so $\frac{1}{2}r(32 - 2r) = 55$

$r^2 - 16r + 55 = 0$, so $(r - 5)(r - 11) = 0$
$r = 5$: $r\theta = 22$, $\theta = \frac{22}{5}$ $r = 11$: $r\theta = 10$, $\theta = \frac{10}{11}$
Answer: $r = 5$, $\theta = \frac{22}{5}$ or $r = 11$, $\theta = \frac{10}{11}$

4 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 137.2 m above the ground for 12 s.
Find the speed of projection and the greatest height reached. Take g = 9.8 m s^{-2}.

By symmetry, the ball takes 6 s to rise from height 137.2 m to its highest point
Speed at height 137.2 m: $v = 9.8 \times 6 = 58.8$ m s^{-1}
Speed of projection: $u^2 = v^2 + 2gh = 3457.44 + 2 \times 9.8 \times 137.2 = 6146.56$

$u = 78.4$. Greatest height = $\frac{u^2}{(2g)} = \frac{6146.56}{19.6} = 313.6$
Answer: 78.4 m s^{-1}, 313.6 m

5 Inverse trigonometric values (Edexcel, OCR, MEI, Eduqas, CIE (inverse trig))

Find the exact value of

$\sin(\sin^{-1}(\frac{4}{5}) + \cos^{-1}(\frac{3}{5}))$.

Let $A = \sin^{-1}(\frac{4}{5})$ and $B = \cos^{-1}(\frac{3}{5})$, both acute

$\sin A = \frac{4}{5}$, $\cos A = \frac{3}{5}$, $\cos B = \frac{3}{5}$, $\sin B = \frac{4}{5}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{4}{5} \times \frac{3}{5} + \frac{3}{5} \times \frac{4}{5}$

Answer: $\mathbf{\frac{24}{25}}$

6 Equations in disguise (substitution) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $x - 7\sqrt{x} + 10 = 0$.

Let $u = \sqrt{x}$: $u^2 - 7u + 10 = 0$.
$(u - 5)(u - 2) = 0$, so $\sqrt{x} = 5$ or $\sqrt{x} = 2$.
Square: $x = 25$ or $x = 4$.
Answer: $x = 4$ or $x = 25$

7 Speed decaying exponentially (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. The velocity of P is v m s^{-1} and its acceleration is $-0.05v$ m s^{-2}. At t = 0, v = 6. Find (a) the time taken for the speed of P to halve, to 3 s.f., (b) the distance P travels in this time.

$\frac{dv}{dt} = -0.05v$: $\int \frac{dv}{v} = -0.05\int dt$, so $\ln v = -0.05t + c$, giving $v = 6\,e^{-0.05t}$

$v = 3$: $e^{-0.05t} = \frac{1}{2}$, so $t = \frac{\ln 2}{0.05}$

$s = \int v\,dt = -\frac{6}{0.05}e^{-0.05t} + c$; distance in this time $= \frac{6}{0.05}(1 - \frac{1}{2})$

Answer: (a) 13.9 s (b) 60 m

8 Alternate turns and sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B take turns, A first, to score; the first to score wins. P(A scores) = $\frac{1}{6}$, P(B scores) = $\frac{1}{2}$ each attempt.

Find P(A wins (a) on attempt 1 or 2, (b) overall).

P(A wins at attempt 1) = $\frac{1}{6}$; to reach attempt 2 both miss first: $(\frac{5}{6})(\frac{1}{2}) = \frac{5}{12}$

(a) $\frac{1}{6} + \frac{5}{12} \times \frac{1}{6} = \frac{17}{72}$

(b) P(A wins) = $\frac{1}{6} + \frac{5}{12} \times \frac{1}{6} + \frac{5}{12}^2 \times \frac{1}{6} + \ldots$

Geometric: first term $\frac{1}{6}$, common ratio $\frac{5}{12}$

$S = \frac{a}{1 - r} = \frac{\frac{1}{6}}{\frac{7}{12}} = \frac{2}{7}$

Answer: (a) $\frac{17}{72}$ (b) $\frac{2}{7}$

9 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{x + 3}{(1 + 3x)(1 + x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $x + 3 \equiv A(1 + x) + B(1 + 3x)$: $A = 4$, $B = -1$.

$f(x) = \frac{4}{1 + 3x} - \frac{1}{1 + x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use $c = 3$, then $c = 1$).

Coefficients of x^j: $4(-3)^j - 1(-1)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) $A = 4$, $B = -1$ (b) $3 - 11x + 35x^2 - 107x^3 + \ldots$; $|x| < \frac{1}{3}$

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}$.

The point P has coordinates $(-4, -5, 8)$.

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} -2 \\ -5 \\ 10 \end{pmatrix} - t\begin{pmatrix} -2 \\ 0 \\ -3 \end{pmatrix}$

$\mathrm{FP} \cdot \mathbf{d} = 0$: $-26 - t(13) = 0$, so $t = -2$

F = (2, 0, 4)

$\mathrm{PF} = \begin{pmatrix} -6 \\ -5 \\ 4 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{77}$

Answer: F = (2, 0, 4), PF = $\sqrt{77}$