Day 127 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from a point O on horizontal ground with speed 63.7 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{5}{12}$. Find the horizontal distance between the two points at which the stone is 11.025 m above the ground. Take g = 9.8 m s^{-2}.

Vertical: $11.025 = 24.5\,t - 4.9t^2$, i.e. $4.9t^2 - 24.5t + 11.025 = 0$

$t = \frac{24.5 \pm \sqrt{(24.5^2 - 19.6 \times 11.025)}}{9.8} = \frac{24.5 \pm 19.6}{9.8}$

The two times differ by 4 s (symmetry about the top of the path)

Horizontal distance = 58.8 × 4

Answer: 235.2 m

2 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 20 seats. Each row has 3 more seats than the row in front. Find

(a) the number of seats in row 60,

(b) the least number of rows for at least 1049 seats.

Arithmetic: a = 20, d = 3.

(a) $u_{60} = 20 + 59 \times 3 = 197$.

(b) $S_n = \frac{n}{2}(40 + 3n - 3) = \frac{n(3n + 37)}{2} \geq 1049$.

$S_{20} = 970 < 1049$, $S_{21} = 1050 \geq 1049$.

Answer: (a) 197 (b) 21 rows

3 Large data set: linear change of units (AQA, Edexcel, OCR, MEI, Eduqas)

Large Data Set, daily mean air temperature (°C), Hurn, October 2015 (n = 31): mean 11, sd 3.9. (F = 1.8C + 32)

(a) Find the mean and sd in °F.

(b) Is a reading of 21.5 more than 2 sd above the mean?

New value = 1.8 × old + 32, so mean = 1.8 × 11 + 32 = 51.80

The constant shifts the mean but not the spread; only the multiplier affects σ

$\sigma = 1.8 \times 3.9 = 7.020$

Outlier cut-off = mean + 2σ = 11 + 7.8 = 18.8 (original units)

21.5 is greater than 18.8, so yes

Answer: (a) mean 51.8, sd 7.02 (b) yes

4 Distance travelled with a change of direction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t seconds its velocity is v m s^{-1}, where

$v = 2t^2 - 18t + 36$ for $0 \leq t \leq 8$

Find (a) the times at which P is at instantaneous rest, (b) the total distance travelled by P in the first 8 seconds.

$v = 2(t - 3)(t - 6)$, so v = 0 when t = 3 and t = 6

v > 0 for 0 < t < 3, v < 0 for 3 < t < 6, v > 0 for t > 6: direction changes at each

$s = \int v\,dt = \frac{2}{3}t^3 - 9t^2 + 36t$

Distances: 0 to 3: 45; 3 to 6: 9; 6 to 8: $\frac{52}{3}$

Total = $45 + 9 + \frac{52}{3}$

Answer: (a) t = 3, t = 6 (b) $\frac{214}{3}$ m

5 Newton cooling model (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A drink cools according to $\theta = 10 + Ae^{-kt}$, where θ °C is the temperature after t minutes. Initially $\theta = 80$, and after 10 minutes $\theta = 45$.
Find A and k, then find when $\theta = 31$ (3 s.f.).

$t = 0$: $10 + A = 80$ so $A = 70$

$t = 10$: $10 + 70e^{-10k} = 45$ so $e^{-10k} = \frac{1}{2}$

$k = \frac{1}{10} \ln 2 = 0.0693$

$70\, e^{-kt} = 21$ so $t = \frac{1}{k} \ln \frac{10}{3}$

$t = 17.4$ minutes

Answer: A = 70, k = 0.0693, t = 17.4 minutes

6 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-6 - 2\sqrt{2}}{4 - 3\sqrt{2}}$ in the form $p + q\sqrt{2}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(4 + 3\sqrt{2})$.
Numerator: $(-6 - 2\sqrt{2})(4 + 3\sqrt{2}) = -36 - 26\sqrt{2}$.
Denominator: $4^2 - 3^2(2) = -2$.
So the fraction $= \frac{-36 - 26\sqrt{2}}{-2}$.

Answer: $18 + 13\sqrt{2}$

7 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (−2, 0, −1), (5, 2, −2), (0, 3, 4).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\mathrm{BA} = \begin{pmatrix} -7 \\ -2 \\ 1 \end{pmatrix}$, $\mathrm{BC} = \begin{pmatrix} -5 \\ 1 \\ 6 \end{pmatrix}$; $\mathrm{BA} \cdot \mathrm{BC} = 39$

$|\mathrm{BA}| = \sqrt{54}$, $|\mathrm{BC}| = \sqrt{62}$

$\cos \mathrm{ABC} = \frac{39}{\sqrt{54}\sqrt{62}} = 0.6740$, so $\mathrm{ABC} = 47.6°$

$|\mathrm{BA} \times \mathrm{BC}| = \left| \begin{pmatrix} -13 \\ 37 \\ -17 \end{pmatrix} \right| = \sqrt{1827}$

Area $= \frac{1}{2}|\mathrm{BA} \times \mathrm{BC}|$

Answer: ABC = 47.6°, area $= \frac{3}{2}\sqrt{203}$

8 Circle through three points (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle passes through P(−5, −10), Q(16, −7) and R(−8, 11). Find its equation.

Centre is on the perpendicular bisector of each chord.
Bisector of PQ: $21x + 3y = 90$.
Bisector of QR: $-24x + 18y = -60$.
Solving: centre (4, 2).
$r^2 = (-5 - 4)^2 + (-10 - 2)^2 = 225$.

Answer: $(x - 4)^2 + (y - 2)^2 = 225$

9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - 2xy + 5y^2 = 64$.
Find the coordinates of the points on C where the tangent is parallel to the x-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -2y}{-2x + 10y}$

Horizontal tangent when $2x + -2y = 0$, so $y = \frac{2x}{2}$

Substitute into the equation of C: $x^2 + (-2xy) + (5y^2) = 64$
Solve for the remaining coordinate
Answer: $(-4, -4)$ and $(4, 4)$

10 Two-tailed binomial test (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A coin is said to land on blue with probability 0.5. In 40 trials it does so 24 times.
Test at the 5% level whether the probability is different from 0.5.

H_0: $p = 0.5$, H_1: $p \neq 0.5$ (two-tailed). $X \sim B(40, 0.5)$
24 is above the mean 20, so use the upper tail: $P(X \geq 24) = 0.134$
Each tail is compared with 2.5%: 0.134 is greater than 2.5%
Do not reject H_0: there is insufficient evidence that the probability is different from 0.5
Answer: Do not reject H_0 (p-value 0.134 against 2.5%)