Day 125 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A tennis ball is projected from a point O on horizontal ground with speed 24.5 m s^{-1} at an angle θ above the horizontal, where $\tan\theta = \frac{3}{4}$. Find the horizontal distance between the two points at which the tennis ball is 6.125 m above the ground. Take g = 9.8 m s^{-2}.

Vertical: $6.125 = 14.7\,t - 4.9t^2$, i.e. $4.9t^2 - 14.7t + 6.125 = 0$

$t = \frac{14.7 \pm \sqrt{(14.7^2 - 19.6 \times 6.125)}}{9.8} = \frac{14.7 \pm 9.8}{9.8}$

The two times differ by 2 s (symmetry about the top of the path)

Horizontal distance = 19.6 × 2

Answer: 39.2 m

2 Equations quadratic in an exponential (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $5^{2x} - 30 \times 5^x + 125 = 0$,
giving each answer exactly and, where not an integer, to 3 s.f.

Let $u = 5^x$: $u^2 - 30u + 125 = 0$

$(u - 25)(u - 5) = 0$, so $5^x = 25$ or $5^x = 5$

Take logarithms: $x = \log_5 5 = 1$; $x = \log_5 25 = 2$

Answer: $x = \log_5 5 = 1$; $x = \log_5 25 = 2$

3 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 7 m s^{-1} from the top of a cliff. The top of the cliff is 87.5 m above the sea. Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-87.5 = 7t - 4.9t^2$, so $4.9t^2 - 7t - 87.5 = 0$

$t = \frac{7 + \sqrt{(7^2 + 19.6 \times 87.5)}}{9.8} = \frac{7 + 42}{9.8} = 5$

Height risen above the cliff top = $\frac{7^2}{(2 \times 9.8)} = 2.50$ m

Total distance = 2 × 2.50 + 87.5

Answer: (a) 5 s (b) 92.5 m

4 Equations with a square root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\sqrt{x + 1} = x - 5$.

Square both sides: $x + 1 = (x - 5)^2$.

$x^2 - 11x + 24 = 0$, so $x = 3$ or $x = 8$.

Check in $\sqrt{x + 1} = x - 5$: $x = 8$ gives $3 \geq 0$ (valid).

$x = 3$ gives right-hand side $-2 < 0$, so it is rejected.

Answer: $x = 8$

5 Histogram: frequency density (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The histogram shows the times, t seconds, of some runners.
Find (a) the total number of runners, (b) an estimate of the number with $9 \leq t < 36$.

Frequency = frequency density × width. Bar 16 to 26: height 1 squares, width 10, frequency 5

So one square of frequency density = 0.5 runners per second

Class frequencies = 28, 12, 5, 12, 4

Total = 61

Estimate = 0.5 × [3(7) + 1(10) + 3(8) + 1(2)] = 28.5

Answer: (a) 61 (b) 28.5

6 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, solve
$\cos 3\theta = -2\cos\theta$ for $0 < \theta < 180°$.

Substitute: $\cos\theta(4\cos^2\theta - 3 + 2) = 0$

$\cos\theta = 0$ or $\cos^2\theta = \frac{1}{4}$

$\cos\theta = 0$ gives 90°

$\cos^2\theta = \frac{1}{4}$ gives 60°, 120°

Answer: θ = 60°, 90°, 120°

7 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x^3 + ax^2 + bx + 20$, where a and b are constants.
$(x - 2)$ is a factor of f(x), and f(x) leaves remainder -16 when divided by $(x - 3)$.
Find a and b, and hence solve f(x) = 0.

f(2) = 0: $16 + 4a + 2b + 20 = 0$, so $4a + 2b + 36 = 0$.
f(3) = -16: $9a + 3b + 74 = -16$.
Solving: $a = -12$, $b = 6$.
$f(x) = 2x^3 - 12x^2 + 6x + 20 = 2(x + 1)(x - 2)(x - 5)$
Answer: a = -12, b = 6; x = -1, 2, 5

8 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = 4$ and

$u_{n+1} = \frac{1}{1 - u_n}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{100} u_r$.

$u_1 = 4$, $u_2 = -\frac{1}{3}$, $u_3 = \frac{3}{4}$, $u_4 = 4$.

$u_4 = u_1$, so the sequence repeats with period 3.

One period sums to $\frac{53}{12}$.

100 = 3 × 33 + 1: 33 full periods plus the first 1 term.

Answer: $\frac{599}{4}$

9 Poisson: sum of two independent variables, conditional (CIE)

X and Y are independent, X ~ Po(2) and Y ~ Po(3).
(a) State the distribution of X + Y.
(b) Given that X + Y = 5, find the probability that X = 3.

X + Y ~ Po(5)

$P(X = 3 \mid X + Y = 5) = \frac{P(X = 3)P(Y = 2)}{P(X + Y = 5)}$

$= \frac{e^{-2}2^3}{3! \times e^{-3}3^2/2!} \div \frac{e^{-5}5^5}{5!}$

= C(5, 3) $(\frac{2}{5})^3$ $(\frac{3}{5})^2$, a binomial probability

= 0.230

Answer: (a) Po(5) (b) 0.230

10 Lines in 3D: intersection and angle (CIE (9709); Further Maths on other boards)

Line l_1: $\mathbf{r} = \begin{pmatrix} 8 \\ -1 \\ 8 \end{pmatrix} + s\begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$,

line l_2: $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

Show whether the lines meet, giving the point of intersection if they do, and find the acute angle between them to 1 d.p.

Equate components: $8 + 2s = 2t$; $-1 - 2s = 1 + t$; $8 + 3s = 6 - t$

First two give $s = -2$, $t = 2$, but these fail in the third equation, so the lines do not meet (skew)

$\cos\varphi = \dfrac{|-1|}{\sqrt{17}\sqrt{6}} = 0.0990$, $\varphi = 84.3°$

Answer: Lines do not meet (skew); angle 84.3°