Day 124 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ax^n$ passes through (9, 216) and (16, 512).
Find the values of a and n.

$216 = a \times 9^n$ and $512 = a \times 16^n$

Divide: $\left(\frac{16}{9}\right)^n = \frac{64}{27}$ so $n = \frac{3}{2}$

$a = 216 \div 9^{\frac{3}{2}} = 8$

Answer: a = 8, n = $\frac{3}{2}$

2 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sqrt{35}$ is irrational, prove by contradiction that $\sqrt{5} + \sqrt{7}$ is irrational.

Assume $\sqrt{5} + \sqrt{7} = r$, where r is rational.
Square both sides: $5 + 7 + 2\sqrt{35} = r^2$.
So $\sqrt{35} = \frac{r^2 - 12}{2}$.
The right-hand side is rational, but $\sqrt{35}$ is irrational.
This is a contradiction, so $\sqrt{5} + \sqrt{7}$ is irrational.
Answer: Proved

3 Discrete random variable with a probability formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$P(X = x) = kx^2$ for x = 1, 2, ..., 5, and 0 otherwise.

(a) Show that $k = \frac{1}{55}$.

(b) Find E(X) and Var(X).
(c) Find Var(3X + 5).

(a) $\Sigma P(X = x) = k(1 + 4 + 9 + 16 + 25) = 55k = 1$, so $k = \frac{1}{55}$

(b) $E(X) = \Sigma x P(X = x) = \frac{1\times1 + 2\times4 + 3\times9 + 4\times16 + 5\times25}{55} = \frac{45}{11}$

$E(X^2) = \frac{1\times1 + 4\times4 + 9\times9 + 16\times16 + 25\times25}{55} = \frac{89}{5}$

$Var(X) = \frac{89}{5} - \left(\frac{45}{11}\right)^2 = \frac{644}{605}$

(c) $Var(3X + 5) = 3^2 Var(X) = 9 \times \frac{644}{605} = \frac{5796}{605}$

Answer: (b) E(X) = $\frac{45}{11}$, Var(X) = $\frac{644}{605}$ (c) $\frac{5796}{605}$

4 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^2 + 5x + 1$ is translated by $\begin{pmatrix}3\\4\end{pmatrix}$

and then a stretch, scale factor $\frac{1}{2}$, parallel to the x-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix}3\\4\end{pmatrix}$: $y = f(x - 3) + 4 = x^2 - x - 1$.

Then a stretch, scale factor $\frac{1}{2}$, parallel to the x-axis: replace x by 2x: $y = f(2x - 3) + 4$.

Expand and simplify to give the quadratic.

Answer: $y = 4x^2 - 2x - 1$

5 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3t^2 - t$, $y = t^3 - t$.
Find the equation of the tangent at the point where $t = 1$,
in the form $y = mx + c$.

$dx/dt = 6t - 1$, $dy/dt = 3t^2 - 1$

At $t = 1$: $dy/dx = \frac{2}{5} = \frac{2}{5}$; point $(2, 0)$

$y - 0 = \frac{2}{5}(x - 2)$

Answer: $\mathbf{y = \frac{2}{5}x - \frac{4}{5}}$

6 Proof by exhaustion (modular) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $n^2 + 3$ is never a multiple of 5, for any integer n.

Every integer is $5q$, $5q + 1$, $5q + 2$, $5q + 3$, $5q + 4$ for some integer q.
Squaring, $n^2 \equiv r^2 \pmod 5$, so:
$0 \to 0$; $1 \to 1$; $2 \to 4$; $3 \to 4$; $4 \to 1$ ($r \to n^2 \bmod 5$)
So $n^2 + 3 \equiv 3$ or 4 or 2 (mod 5).
No residue is 0, so $n^2 + 3$ is never a multiple of 5.
Answer: Proved

7 Force up a rough slope: find weight and friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight W N is on a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{12}{13}$. A force P N acts up a line of greatest slope; the particle stays at rest for $88 \le P \le 176$.
Find (a) W, (b) the coefficient of friction, (c) the acceleration when $P = 352$. Take $g = 9.8$ m s^{-2}.

Limiting friction acts down the plane at $P = 176$ and up the plane at $P = 88$
$P_{max} = W\sin\alpha + F$ and $P_{min} = W\sin\alpha - F$, where $F = \mu W\cos\alpha$

Adding: $2W\sin\alpha = 264$, so $W = \frac{264}{2 \times \frac{12}{13}} = 143$

Subtracting: $2F = 88$, so $F = 44 = \mu \times 143 \times \frac{5}{13}$, giving $\mu = 0.8$

$P = 352$: net force up the plane $= 352 - 176 = 176$, and mass $= \frac{143}{9.8}$

Answer: (a) 143 N (b) 0.8 (c) 12.1 m s^{-2}

8 Area under a parametric curve (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3\sin t$, $y = 2\cos t$,
for $0 \le t \le \frac{\pi}{2}$. Find the exact area bounded by the curve
and the coordinate axes.

$dx/dt = 3\cos t$

Area $= \int_0^{\pi/2} 2\cos t \times 3\cos t\, dt = 6\int_0^{\pi/2} \cos^2 t\, dt$

$\cos^2 t = \frac{1}{2}(1 + \cos 2t)$: integral $= 6 \times \frac{\pi}{4}$

Answer: $\mathbf{\frac{3\pi}{2}}$

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9 Acceleration given, find when P returns to O (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P moves on a straight line. At time t = 0, P passes through the fixed point O with velocity 12 m s^{-1}. At time t seconds, P has acceleration (3t − 11) m s^{-2}.
Find (a) the times at which P is at O again, (b) the greatest distance of P from O before it first returns to O.

$v = \int a\,dt = 1.5t^2 - 11t + 12$ ($v = 12$ when $t = 0$)
$s = \int v\,dt = 0.5t^3 - 5.5t^2 + 12t$ ($s = 0$ when $t = 0$)
$s = 0.5t(t - 3)(t - 8)$, so P is at O when $t = 3$ and $t = 8$

Greatest distance when $v = 0$: $1.5t^2 - 11t + 12 = 0$, so $t = \frac{4}{3}$ (the smaller root)

$|s| = 0.5 \times \frac{4}{3} \times \frac{5}{3} \times \frac{20}{3}$

Answer: (a) t = 3, t = 8 (b) $\frac{200}{27}$ (= 7.41) m

10 Normal: find mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The scores in an aptitude test are normally distributed. 20% are less than 55 and 15% are greater than 91.
Find the mean and standard deviation, each correct to 3 significant figures.

Let $X \sim N(\mu, \sigma^2)$. $P(X < 55) = 0.20$ gives $\frac{55 - \mu}{\sigma} = -0.8416$

$P(X > 91) = 0.15$ gives $\frac{91 - \mu}{\sigma} = 1.0364$

Subtract: $\frac{36}{\sigma} = 1.0364 - (-0.8416) = 1.8781$

$\sigma = 19.17$
$\mu = 55 - (-0.8416)(19.17) = 71.13$
Answer: μ = 71.1, σ = 19.2