Day 123 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = -\frac{1}{2}u_n + 24$, $u_1 = 10$.

(a) Find the limit L of the sequence.
(b) Find the least n such that $|u_n - L| < 0.001$.

(a) At the limit $L = -\frac{1}{2}L + 24$, so $L(1 - (-\frac{1}{2})) = 24$.

$L = 16$.

(b) $u_n - L = (u_1 - L)((-\frac{1}{2}))^{n-1} = (-6)((-\frac{1}{2}))^{n-1}$.

Need $6 \times (\frac{1}{2})^{n-1} < 0.001$, so $n - 1 > 12.551$.

The least n is 14.

Answer: (a) L = 16 (b) n = 14

2 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 21 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 21 m horizontally and 16 m vertically from O.

Find the two possible values of $\tan\theta$, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 21\cos\theta \times t$. Vertical: $y = 21\sin\theta \times t - 4.9t^2$

Eliminating t: $y = x\tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 21^2}{2 \times 21^2} = 4.9$, so $16 = 21\tan\theta - 4.9(1 + \tan^2\theta)$

$4.9\tan^2\theta - 21\tan\theta + 20.9 = 0$

$\tan\theta = \frac{11}{7}$ or $\frac{19}{7}$, so $\theta = 57.5°$ or $69.8°$

Answer: $\tan\theta = \frac{11}{7}$ or $\frac{19}{7}$; $\theta = 57.5°$ or $69.8°$

3 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = x^2 + 1$. Also $fg(x) = -3x^2 + 6$, where f is a linear function. (a) Find f(x).
(b) Solve $gf(x) = 10$.

(a) Let $f(x) = ax + b$. $fg(x) = a(x^2 + 1) + b = -3x^2 + 6$.
So $a = -3$ and $-3 + b = 6$, giving $b = 9$.
$f(x) = -3x + 9$.
(b) $gf(x) = (-3x + 9)^2 + 1 = 10$.
$(-3x + 9)^2 = 9$, so $-3x + 9 = \pm 3$.
$x = 2$ or $x = 4$.

Answer: (a) f(x) = −3x + 9 (b) x = 2 or x = 4

4 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $e^x + 10\,e^{-x} = 7$,
giving your answers exactly in terms of natural logarithms.

Multiply by e^x: $e^{2x} - 7e^x + 10 = 0$
$(e^x - 5)(e^x - 2) = 0$
$e^x = 2$ or $e^x = 5$

Answer: x = ln 2 and x = ln 5

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5 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 3 and variance 2.7.
(a) Find n and p.
(b) Find P(X = 3), correct to 3 significant figures.

$np = 3$ and $np(1 - p) = 2.7$

Divide: $1 - p = 2.7 \div 3 = \frac{9}{10}$, so $p = \frac{1}{10}$

$n = 3 \div \frac{1}{10} = 30$

$P(X = 3) = C(30, 3)\left(\frac{1}{10}\right)^3\left(\frac{9}{10}\right)^{27} = 0.236$

Answer: (a) n = 30, p = $\frac{1}{10}$ (b) 0.236

6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = t^2$, $y = 2t^3 - 24t$.
Find the coordinates of the points on the curve where the gradient is 9.

$dx/dt = 2t$, $dy/dt = 6t^2 - 24$, so $dy/dx = \frac{6t^2 - 24}{2t}$

Set equal to 9: $6t^2 - 18t - 24 = 0$
$t = -1$ or $t = 4$
Answer: (1, 22) and (16, 32)

7 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{24}{2x^2 - 16x + 34}$ for all real x.
Find the greatest value of f(x), the value of x at which it occurs, and the range of f.

$2x^2 - 16x + 34 = 2(x - 4)^2 + 2$.
$(x - 4)^2 \geq 0$, so the denominator is at least 2, with equality at $x = 4$.
The denominator is always positive, so $f(x) > 0$.
Greatest $f = \frac{24}{2} = 12$ at $x = 4$.

Answer: greatest value 12 at x = 4; range $0 < f(x) \leq 12$

8 Shaded normal curve and the standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The pdf of $X \sim N(97, \sigma^2)$ is shown; the shaded area is symmetrical.
(a) Find σ. (b) Find P(X > 105).

The area 0.8 leaves 0.200 in total in the tails, 0.100 in each
$P(X < 93) = 0.100$ so $\frac{93 - 97}{\sigma} = -1.2816$

$\sigma = \frac{4}{1.2816} = 3.121$
$P(X > 105) = P(Z > 2.563) = 0.00519$
Answer: (a) 3.12 (b) 0.00519

9 Ladder against a smooth wall: how high can a person climb (AQA, Edexcel, OCR, MEI)

A uniform ladder AB of length 6 m and weight 200 N has A on rough ground ($\mu = 0.6$) and B against a smooth vertical wall. It makes angle θ with the ground, $\tan\theta = \frac{4}{3}$.

A man of weight 500 N climbs the ladder. How far along the ladder can he go before it slips?

Wall reaction S horizontal, ground reaction R vertical, friction F = S. Moments about A: $S \times L \sin\theta = W \times \frac{L}{2} \cos\theta$

So $S = \frac{W}{2\tan\theta}$. Unloaded, R = W: need $\frac{W}{2\tan\theta} \leq \mu W$, so $\mu \geq \frac{1}{2 \times \frac{4}{3}} = 0.375$

With the man at distance x: R = 700, and moments about A give $S L \sin\theta = (200 \times 3 + 500x)\cos\theta$

Limiting: $S = \mu R = 0.6 \times 700 = 420$, so $420 \times 6 \times \frac{4}{3} = 200 \times 3 + 500x$

$500x = 3360 - 600$, so $x = 5.52$

Answer: 5.52 m

10 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A rectangle has its base on the x-axis and its upper vertices on the curve $y = 12 - x^2$, symmetric about the y-axis.

Find the maximum area of the rectangle, with justification.

Base 2x, height $12 - x^2$: $A = 2x(12 - x^2) = 24x - 2x^3$

$dA/dx = 24 - 6x^2 = 0$, so $x = 2$ ($x > 0$)

$d^2A/dx^2 = -12x < 0$: maximum

$A = 2 \times 2 \times 8$

Answer: 32