Day 122 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ax^n$ passes through (1, 5) and (9, 135).
Find the values of a and n.

$5 = a \times 1^n$ and $135 = a \times 9^n$

Divide: $9^n = 27$ so $n = \frac{3}{2}$

$a = 5 \div 1^{\frac{3}{2}} = 5$

Answer: a = 5, n = $\frac{3}{2}$

2 Histogram: frequency density (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The histogram shows the times, t seconds, of some runners.
Find (a) the total number of runners, (b) an estimate of the number with $11 \leq t < 30$.

Frequency = frequency density × width. Bar 0 to 8: height 4 squares, width 8, frequency 64
So one square of frequency density = 2 runners per second
Class frequencies = 64, 16, 32, 12, 80
Total = 204
Estimate = 2 × [2(1) + 2(8) + 1(6) + 4(4)] = 80
Answer: (a) 204 (b) 80

3 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = x^2 - 2x - 4$
for $x \leq k$. Given that k is the greatest value for which
f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = (x - 1)^2 - 5$.
The vertex is at x = 1, so k = 1 (f is one-one on one side of it).
Let $y = f(x)$: $y + 5 = (x - 1)^2$, so $(x - 1) = \pm\sqrt{y + 5}$.
As $x \leq 1$, take the negative root: $x = 1 - \sqrt{y + 5}$.
Range of f = domain of f^{-1}: $f(1) = -5$, so $x \geq -5$.
Answer: k = 1; $f^{-1}(x) = 1 - \sqrt{x + 5}$; $x \geq -5$

4 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is acute. $\cos A = \frac{3}{5}$ and $\cos B = \frac{12}{13}$.
Find the exact values of sin(A + B) and cos(A + B).

$\cos A = \frac{3}{5}$, $\sin A = \frac{4}{5}$; $\sin B = \frac{5}{13}$, $\cos B = \frac{12}{13}$

$\sin(A + B) = \sin A \cos B + \cos A \sin B = \frac{63}{65}$

$\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{16}{65}$

Answer: $\sin(A + B) = \frac{63}{65}$, $\cos(A + B) = \frac{16}{65}$

5 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = -3$ and

$u_{n+1} = \frac{2u_n + 1}{u_n - 2}$.

Show that the sequence is periodic and find

$\sum_{r=1}^{60} u_r$.

$u_1 = -3$, $u_2 = 1$, $u_3 = -3$.

$u_3 = u_1$, so the sequence repeats with period 2.

One period sums to -2.

$60 = 2 \times 30 + 0$: 30 full periods plus the first 0 terms.

Answer: -60

6 Exponential growth and decay models (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A radioactive sample has mass $m = m_0 e^{-kt}$ grams, t in days.

Initially m = 200 and after 2 days m = 100.

Find k, and the time for the mass to fall to 10 grams, to 3 s.f.

$m_0 = 200$. $200\, e^{-2k} = 100$ so $e^{-2k} = \frac{1}{2}$

$k = \frac{1}{2} \ln 2 = 0.347$

$200\, e^{-kt} = 10$ so $e^{kt} = 20$, $t = \frac{1}{k} \ln 20$

t = 8.64 days

Answer: k = 0.347, t = 8.64 days

7 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(\mathbf{i} + 40\mathbf{j})$ m s^{-1} and $(6\mathbf{i} + 28\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).

Both are still in flight after 2 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $1t\mathbf{i} + (40t - 4.9t^2)\mathbf{j}$; position of B: $6t\mathbf{i} + (28t - 4.9t^2)\mathbf{j}$

Subtracting, the $4.9t^2$ terms cancel: $AB = t(-5\mathbf{i} + 12\mathbf{j})$

After 2 s: $AB = (-10\mathbf{i} + 24\mathbf{j})$

Distance $= \sqrt{(10^2 + 24^2)} = 26$

The direction of AB is that of $(-5\mathbf{i} + 12\mathbf{j})$, which does not depend on t: tan angle $= \frac{12}{5}$

Answer: (a) 26 m (b) 67.4°

8 Sum of two independent copies of a discrete variable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X takes values 1, 2, 3, 4 with $P(X = x) = k(x + 2)$. X_1 and X_2 are independent copies of X.

(a) Find k.

(b) Find $P(X_1 + X_2 = 6)$.

(c) Find $P(X_1 = X_2)$ and $E(X_1 + X_2)$.

(a) $\Sigma P = k(3 + 4 + 5 + 6) = 18$ k = 1, so $k = \frac{1}{18}$

(b) Ways to total 6: (2,4), (3,3), (4,2)

$P = \frac{2}{9} \times \frac{1}{3} + \frac{5}{18} \times \frac{5}{18} + \frac{1}{3} \times \frac{2}{9} = \frac{73}{324}$

(c) $P(X_1 = X_2) = \Sigma [P(X = x)]^2 = \frac{43}{162}$

$E(X) = \frac{25}{9}$, so $E(X_1 + X_2) = 2 \times \frac{25}{9} = \frac{50}{9}$

Answer: (a) $k = \frac{1}{18}$ (b) $\frac{73}{324}$ (c) $\frac{43}{162}$, $\frac{50}{9}$

9 Horizontal force on a rough slope: range of values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight 160 N rests on a rough plane inclined at α to the horizontal, where $\tan\alpha = \frac{4}{3}$ and $\mu = 0.25$.

A horizontal force H N, in the vertical plane through a line of greatest slope, holds it in equilibrium. Find the range of possible values of H.

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$. Normal reaction: $R = 160\cos\alpha + H\sin\alpha$

About to slide down: $H\cos\alpha + \mu R = 160\sin\alpha$, so $H(\frac{3}{5} + 0.25 \times \frac{4}{5}) = 160(\frac{4}{5} - 0.25 \times \frac{3}{5})$

$H_{min} = 130$

About to slide up: $H\cos\alpha = 160\sin\alpha + \mu R$, so $H(\frac{3}{5} - 0.25 \times \frac{4}{5}) = 160(\frac{4}{5} + 0.25 \times \frac{3}{5})$

$H_{max} = 380$

Answer: $130 \leq H \leq 380$

10 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{7x - 1}{(1 - 3x)(1 - x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $7x - 1 \equiv A(1 - x) + B(1 - 3x)$: $A = 2$, $B = -3$.

$f(x) = \frac{2}{1 - 3x} - \frac{3}{1 - x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use $c = -3$, then $c = -1$).

Coefficients of x^j: $2(3)^j - 3(1)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) $A = 2$, $B = -3$ (b) $-1 + 3x + 15x^2 + 51x^3 + \ldots$; $|x| < \frac{1}{3}$