Day 121 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 4 red counters and some white counters, n in total.

Two are taken without replacement. P(at least one red) = $\frac{14}{15}$. Find n.

P(neither red) = $\frac{(n-4)}{n} \times \frac{(n-5)}{(n-1)} = \frac{1}{15}$

$\frac{1}{15}\, n(n-1) = (n-4)(n-5)$

Rearranging gives a quadratic in n whose roots are n = 6 and one other value

The other root is $\frac{25}{7}$ so is rejected (not a valid number of counters)

n = 6

Answer: n = 6

2 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = 6xy$, and y = 3 when x = 0.

Solve the differential equation and find y when x = 1.

$\int \frac{1}{y}\, dy = \int 6x\, dx$, so $\ln y = 6x^2/2 + c = 3x^2 + c$

x = 0, y = 3: c = ln 3, so $y = 3\, e^{3x^2}$

x = 1: exponent = 3

Answer: $y = 3\, e^{3x^2}$; $y = 3e^3$

3 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 308.7 m above the ground for 2 s.

Find the speed of projection and the greatest height reached. Take g = 9.8 m s^{-2}.

By symmetry, the ball takes 1 s to rise from height 308.7 m to its highest point

Speed at height 308.7 m: $v = 9.8 \times 1 = 9.8$ m s^{-1}

Speed of projection: $u^2 = v^2 + 2gh = 96.04 + 2 \times 9.8 \times 308.7 = 6146.56$

u = 78.4. Greatest height = $\frac{u^2}{(2g)} = \frac{6146.56}{19.6} = 313.6$

Answer: 78.4 m s^{-1}, 313.6 m

4 Normal: unknown standard deviation then a second probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The scores in an aptitude test are modelled by $N(48, \sigma^2)$. The probability of a value below 44 is 0.30.

(a) Find σ.

(b) Find the probability of a value above 61.

P(X < 44) = 0.30 so $Z = \frac{44-48}{\sigma} = -0.5244$

$\sigma = \frac{-4}{-0.5244} = 7.628$

$z = \frac{61-48}{7.628} = 1.704$

P(X > 61) = P(Z > 1.704) = 0.0442

Answer: (a) 7.63 (b) 0.0442

5 Proof by deduction (divisibility) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove that $(n + 5)^3 - n^3 - 125$ is a multiple of 15 for every integer n.

$(n + 5)^3 = n^3 + 15n^2 + 75n + 125$.

So the expression = $15n^2 + 75n$

= 15n(n + 5), a multiple of 15.

Answer: Proved

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6 Force up a rough slope: find weight and friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight W N is on a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{4}{5}$. A force P N acts up a line of greatest slope; the particle stays at rest for $70.3 \leq P \leq 81.7$.

Find (a) W, (b) the coefficient of friction, (c) the acceleration when P = 163.4. Take g = 9.8 m s^{-2}.

Limiting friction acts down the plane at P = 81.7 and up the plane at P = 70.3

$P_{max} = W\sin\alpha + F$ and $P_{min} = W\sin\alpha - F$, where $F = \mu W\cos\alpha$

Adding: $2W\sin\alpha = 152$, so $W = \frac{152}{2 \times \frac{4}{5}} = 95$

Subtracting: $2F = 11.4$, so $F = 5.7 = \mu \times 95 \times \frac{3}{5}$, giving $\mu = 0.1$

P = 163.4: net force up the plane = 163.4 − 81.7 = 81.7, and mass = $\frac{95}{9.8}$

Answer: (a) 95 N (b) 0.1 (c) 8.43 m s^{-2}

7 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^{\pi/2} x\cos x\,dx$

Let u = x and dv/dx = cos x

Integral = [uv] ∓ ... using v = sin x

Integral = $[\frac{x\sin 1x}{1} + \frac{\cos 1x}{1}]$

Answer: $-1 + \frac{1}{2}\pi$

8 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{-16x + 6}{(1 - 3x)(1 - 2x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $-16x + 6 \equiv A(1 - 2x) + B(1 - 3x)$: A = 2, B = 4.

$f(x) = \frac{2}{1 - 3x} + \frac{4}{1 - 2x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use c = −3, then c = −2).

Coefficients of x^j: $2(3)^j + 4(2)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) A = 2, B = 4 (b) $6 + 14x + 34x^2 + 86x^3 + \ldots$; $|x| < \frac{1}{3}$

9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - 2xy + 3y^2 = 24$.

Find the coordinates of the points on C where the tangent is parallel to the y-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -2y}{-2x + 6y}$

Vertical tangent when $-2x + 6y = 0$, so $x = \frac{6y}{2}$

Substitute into the equation of C: $x^2 + (-2xy) + (3y^2) = 24$

Solve for the remaining coordinate

Answer: (−6, −2) and (6, 2)

10 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = \sqrt{-x^2 - 2x}$.

Find the largest possible domain of g, and the corresponding range.

Need $-x^2 - 2x \geq 0$, i.e. $(x + 2)x \leq 0$.

Domain: $-2 \leq x \leq 0$.

Complete the square: $-x^2 - 2x = 1 - (x + 1)^2$.

Greatest value of $g = \sqrt{1} = 1$, and least value is 0.

Answer: $-2 \leq x \leq 0$; $0 \leq g(x) \leq 1$