Day 120 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Projectile on level ground (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from a point O on horizontal ground with speed 49 m s^{-1} at an angle α above the horizontal, where $\tan\alpha = \frac{4}{3}$. Find (a) the time of flight, (b) the range, (c) the greatest height. Take g = 9.8 m s^{-2}.

Components: horizontal 29.4, vertical 39.2 (as $\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$)

Time of flight: $0 = 39.2\,t - 4.9t^2$, so $t = \frac{39.2}{4.9} = 8$

Range = $29.4 \times 8 = 235.2$

Greatest height: $0 = 39.2^2 - 2 \times 9.8 \times H$, so $H = \frac{1536.64}{19.6} = 78.4$

Answer: (a) 8 s (b) 235.2 m (c) 78.4 m

2 Normal: find mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The heights of plants, in cm are normally distributed. 5% are less than 42 and 30% are greater than 66. Find the mean and standard deviation, each correct to 3 significant figures.

Let $X \sim N(\mu, \sigma^2)$. $P(X < 42) = 0.05$ gives $\frac{42 - \mu}{\sigma} = -1.6449$

$P(X > 66) = 0.30$ gives $\frac{66 - \mu}{\sigma} = 0.5244$

Subtract: $\frac{24}{\sigma} = 0.5244 - (-1.6449) = 2.1693$

$\sigma = 11.06$

$\mu = 42 - (-1.6449)(11.06) = 60.20$

Answer: μ = 60.2, σ = 11.1

3 Trapezium rule and over/under estimates (AQA, Edexcel, OCR, MEI, Eduqas (trapezium rule))

Use the trapezium rule with 6 strips to estimate

$\int_1^2 \frac{1}{\sqrt{1+x}}\,dx$, to 4 s.f.

Is it an overestimate or an underestimate? Explain.

$h = \frac{1}{6}$; ordinates at $x = 1, \frac{7}{6}, \frac{4}{3}, \frac{3}{2}, \ldots, 2$

Sum = $\frac{h}{2}[y_0 + y_6 + 2(y_1 + \ldots + y_5)] = 0.6359$

f''(x) > 0 on the interval, so the curve is convex and the trapezia lie above the curve

Answer: 0.6359; overestimate

4 Binomial: mean and variance given (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A binomial random variable X has mean 16 and variance 9.6.

(a) Find n and p.

(b) Find P(X = 16), correct to 3 significant figures.

$np = 16$ and $np(1 - p) = 9.6$

Divide: $1 - p = 9.6 \div 16 = \frac{3}{5}$, so $p = \frac{2}{5}$

$n = 16 \div \frac{2}{5} = 40$

$P(X = 16) = C(40, 16)\,(\frac{2}{5})^{16}\,(\frac{3}{5})^{24} = 0.128$

Answer: (a) n = 40, p = $\frac{2}{5}$ (b) 0.128

5 Pair of inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the set of values of x for which both
$x^2 + 2x - 3 > 0$ and $-x + 3 \leq 6$.

Quadratic: $(x + 3)(x - 1) > 0$ gives $x < -3$ or $x > 1$.
Linear: $-x + 3 \leq 6$ gives $x \geq -3$. (the inequality reverses as we divide by a negative)
Both must hold: take the overlap of the two sets.
Answer: $x > 1$

6 Stationary points of x^n times a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 3x^2\sqrt{20 - x}$, for $0 < x < 20$, has one stationary point. Find its exact coordinates and determine its nature.

$\frac{dy}{dx} = \frac{3x(80 - 5x)}{2\sqrt{20 - x}}$
$dy/dx = 0$: $80 - 5x = 0$, so $x = 16$
$y = 3 \times 16^2 \times 2$
Gradient changes from positive to negative: a maximum
Answer: (16, 1536), maximum

7 Circle touching the axes (diagram) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle in the diagram touches both axes and passes through P. Find the equation of the circle, giving it in the form $x^2 + y^2 + ax + by + c = 0$.

Touches both axes in the first quadrant: centre (r, r), radius r.
P(8, 9) lies on the circle: $(8 - r)^2 + (9 - r)^2 = r^2$.
$r^2 - 34r + 145 = 0$, so $r = 5$ or $r = 29$.
The diagram shows P on the smaller circle, so $r = 5$.
$(x - 5)^2 + (y - 5)^2 = 25$, then expand.
Answer: $x^2 + y^2 - 10x - 10y + 25 = 0$

8 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_5 12$ is irrational.

Assume $\log_5 12 = \frac{m}{n}$, m, n positive integers.

Then $5^{\frac{m}{n}} = 12$, so $5^m = 12^n$.
Prime factors of 5^m and 12^n must be identical,
but 2 divides 12^n but 2 does not divide 5^m.
This is a contradiction, so $\log_5 12$ is irrational.
Answer: Proved

9 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(5\mathbf{j})$ km relative to a fixed origin and B is at $(-15\mathbf{i})$ km. A has velocity $(-3\mathbf{i} + 2\mathbf{j})$ km h^{-1} and B has velocity $(\mathbf{i} + 5\mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-15\mathbf{i} - 5\mathbf{j}) + t(4\mathbf{i} + 3\mathbf{j})$
$|\mathbf{d}|^2 = (4t - 15)^2 + (3t - 5)^2 = 25t^2 - 150t + 250$
Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-75}{25} = 3$
$\mathbf{d} = (-15\mathbf{i} - 5\mathbf{j}) + 3(4\mathbf{i} + 3\mathbf{j}) = (-3\mathbf{i} + 4\mathbf{j})$, so least distance = 5
Answer: (a) t = 3 h (b) 5 km

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10 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{x^2 + 12x + 22}{(x + 3)(x + 2)} \equiv 1 + \frac{A}{x + 3} + \frac{B}{x + 2}$.

(b) Hence find the exact value of $\int_0^1 \frac{x^2 + 12x + 22}{(x + 3)(x + 2)}\,dx$.

Numerator = $(x + 3)(x + 2) + A(x + 2) + B(x + 3)$
$x = -3$ gives $A = 5$; $x = -2$ gives $B = 2$
Integral = $[x + 5\ln(x + 3) + 2\ln(x + 2)]$
Substitute limits 0 and 1
Answer: A = 5, B = 2; 1 + 8ln 2 − 3ln 3