Day 119 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 8 m and mass 30 kg rests on supports at C and D, where AC = 3 m and DB = 2 m. A girl of mass 50 kg stands on it at distance x m from A.
Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(2) = 50(x - 6)$, so $x = 6 + \frac{30 \times 2}{50} = 7.2$

Tilting about C (girl before C): $30(1) = 50(3 - x)$, so $x = 3 - \frac{30 \times 1}{50} = 2.4$

Answer: $2.4 \leq x \leq 7.2$

2 Reading a log graph (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = ax^n$. The graph of lg y against lg x is the straight line shown.
Find a and n, giving a to 3 s.f.

$\lg y = \lg a + n \lg x$. The intercept is $\lg a = 1.2$, so $a = 10^{1.2} = 15.85$

n = gradient $= \frac{8.7 - 1.2}{3} = 2.5$

Answer: $a = 15.8$, $n = 2.5$

3 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 269.5 m above the ground for 6 s.
Find the speed of projection and the greatest height reached. Take $g = 9.8$ m s^{-2}.

By symmetry, the ball takes 3 s to rise from height 269.5 m to its highest point

Speed at height 269.5 m: $v = 9.8 \times 3 = 29.4$ m s^{-1}

Speed of projection: $u^2 = v^2 + 2gh = 864.36 + 2 \times 9.8 \times 269.5 = 6146.56$

$u = 78.4$. Greatest height $= \frac{u^2}{(2g)} = \frac{6146.56}{19.6} = 313.6$

Answer: 78.4 m s^{-1}, 313.6 m

4 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 3^x - x$ has one stationary point.
Find its coordinates to 3 s.f. and show that it is a minimum.

$dy/dx = 3^x \ln 3 - 1 = 0$

$3^x = \frac{1}{\ln 3}$, $x = \frac{1}{\ln 3} \ln(\frac{1}{\ln 3}) = -0.0856$

$d^2y/dx^2 = 3^x(\ln 3)^2 > 0$ for all x: minimum

$y = \frac{1}{\ln 3} - 1 \times x = 0.996$

Answer: $(-0.0856, 0.996)$

5 Indices: simultaneous equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously, giving integer x and y:

$2^x \times 16^y = 32$

$27^x \div 9^y = 3$

Write each side with prime bases: $32 = 2^5$, $3 = 3^1$.

First: $2^{1x} \times 2^{4y} = 2^5$, so $1x + 4y = 5$.

Second: $3^{3x - 2y} = 3^1$, so $3x - 2y = 1$.

Solving the pair of linear equations: $x = 1$, $y = 1$.

Answer: $x = 1$, $y = 1$

6 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^{\pi/6} x \cos 3x \, dx$

Let u = x and dv/dx = cos 3x

Integral = [uv] ∓ ... using v = $\frac{1}{3}\sin 3x$

Integral = $[\frac{x \sin 3x}{3} + \frac{\cos 3x}{9}]$

Answer: $-\frac{1}{9} + \frac{1}{18}\pi$

7 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A circle has its centre in the first quadrant and touches both the x-axis and the y-axis. It passes through the point (2, 9). Find the two possible equations of the circle.

Centre (r, r), radius r: $(x - r)^2 + (y - r)^2 = r^2$.

Substitute (2, 9): $(2 - r)^2 + (9 - r)^2 = r^2$.

$r^2 - 22r + 85 = 0$.

$(r - 5)(r - 17) = 0$, so r = 5 or r = 17.

Answer: $(x - 5)^2 + (y - 5)^2 = 25$ or $(x - 17)^2 + (y - 17)^2 = 289$

8 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 4 red, 6 blue and 2 green counters. A player takes two without replacement. Two reds win £3, two blues win £7, anything else wins nothing.

(a) Find the expected winnings.

(b) The player pays £1 to play. Find the expected profit.

P(RR) = $\frac{4}{12} \times \frac{3}{11} = \frac{1}{11}$

P(BB) = $\frac{6}{12} \times \frac{5}{11} = \frac{5}{22}$

(a) E = $3 \times \frac{1}{11} + 7 \times \frac{5}{22} = \frac{41}{22}$

(b) Expected profit = $\frac{41}{22} - 1 = \frac{19}{22}$

Answer: (a) £1.86 (b) £0.864

9 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 4x^2 + 7x - 4$.

Show that (x − 1) is a factor of f(x), and that the equation f(x) = 0 has exactly one real root.

f(1) = 1 − 4 + 7 − 4 = 0, so (x − 1) is a factor.

$f(x) = (x - 1)(x^2 - 3x + 4)$.

For $x^2 - 3x + 4 = 0$: $b^2 - 4ac = 9 - 16 = -7 < 0$.

No real roots from the quadratic, so the only real root is x = 1.

Answer: Proved; the only real root is x = 1

10 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{-2x} \cos 2x$. Show that $dy/dx = e^{-2x}(-2 \cos 2x - 2 \sin 2x)$.

Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (-2)e^{-2x} \cos 2x - e^{-2x} \times 2 \sin 2x$

dy/dx = 0 requires $-2 \cos 2x = 2 \sin 2x$

$\tan 2x = -1$, smallest positive 2x = 2.3562

x = 1.18

Second derivative test (or gradient change): a minimum

Answer: x = 1.18, minimum