Day 118 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 2x^3 + ax^2 + bx + 40$, where a and b are constants.
$(x - 4)$ is a factor of f(x), and f(x) leaves remainder 48 when divided by $(x - 1)$.
Find a and b, and hence solve $f(x) = 0$.

$f(4) = 0$: $128 + 16a + 4b + 40 = 0$, so $16a + 4b + 168 = 0$.
$f(1) = 48$: $a + b + 42 = 48$.
Solving: $a = -16$, $b = 22$.
$f(x) = 2x^3 - 16x^2 + 22x + 40 = 2(x + 1)(x - 4)(x - 5)$
Answer: $a = -16$, $b = 22$; $x = -1, 4, 5$

2 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from level ground with horizontal and vertical velocity components 19.6 and 29.4 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 19.6$
Moving upwards: $29.4 - 9.8t = 19.6$, so $t = 1$
Moving downwards: $29.4 - 9.8t = -19.6$, so $t = 5$
Horizontal distance $= 19.6 \times (5 - 1)$
Answer: (a) 1 s and 5 s (b) 78.4 m

3 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution $u = \ln x$ to find the exact value of

$$\int_1^{e^2} \frac{(\ln x)^2}{x}\,dx$$

$du = \frac{1}{x}\,dx$; limits $x = 1, e^2$ become $u = 0, 2$

Integral $= \int_0^2 u^2\,du$

Evaluate the limits

Answer: $\frac{8}{3}$

4 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^2 - x - 4$ is translated by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$
and then a stretch, scale factor 2, parallel to the x-axis.
Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$: $y = f(x - 3) - 2 = x^2 - 7x + 6$.

Then a stretch, scale factor 2, parallel to the x-axis: replace x by $\frac{1}{2}x$: $y = f(\frac{1}{2}x - 3) - 2$.

Expand and simplify to give the quadratic.

Answer: $y = \frac{1}{4}x^2 - \frac{7}{2}x + 6$

5 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $9 \sin\theta + 12 \cos\theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $9 \sin\theta + 12 \cos\theta = -8$ for $0 \le \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{9^2 + 12^2} = 15$ and $\tan\alpha = \frac{12}{9}$, so $\alpha = 53.1°$

So $9 \sin\theta + 12 \cos\theta \equiv 15 \sin(\theta + 53.1°)$

$\sin(\theta + \alpha) = \frac{-8}{15}$

$\theta + \alpha = -32.2°$ or $212.2°$ (using $\alpha = 53.1°$)
Adding or subtracting multiples of 360° to give $0 \le \theta < 360°$
Answer: (a) $R = 15$, $\alpha = 53.1°$ (b) $\theta = 159.1°, 274.6°$

6 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(12, p). It is given that $P(X = 2) = \frac{11}{8} \times P(X = 1)$.

(a) Find the value of p.
(b) Find P(X = 0), correct to 3 significant figures.

$\frac{132}{2} p^2(1 - p)^{10} = \frac{11}{8} \times 12\, p(1 - p)^{11}$

Divide by $p(1 - p)^{10}$: $5.5\, p = \frac{11}{8}(1 - p)$

$(5.5 + \frac{11}{8})p = \frac{11}{8}$

$p = \frac{1}{5}$

$P(X = 0) = (1 - p)^{12} = (\frac{4}{5})^{12} = 0.0687$

Answer: (a) $p = \frac{1}{5}$ (b) 0.0687

7 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $6\text{cosec}^2\theta + 2\cot\theta - 10 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $6\cot^2\theta + 2\cot\theta - 4 = 0$
$2(3\cot\theta - 2)(\cot\theta + 1) = 0$

$\cot\theta = \frac{2}{3}$ or $\cot\theta = -1$

$\tan\theta = \frac{3}{2}$ or -1, so $\theta = 56.3°$ and $135.0°$ $(+180°)$

Answer: θ = 56.3°, 135.0°, 236.3°, 315.0°

8 Tree diagram with an unknown probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tree diagram shows two games played in turn.

Given the second is a W, P(first was a W) = $\frac{8}{9}$. Find x.

P(W then W) = 0.8x, P(second W) = $0.8x + 0.2(1 - x)$

P(first W | second W) = $\frac{0.8x}{0.8x + 0.2(1 - x)} = \frac{8}{9}$

$0.8x = \frac{8}{9}(0.8x + 0.2 - 0.2x)$

Solve the linear equation for x: $x = \frac{2}{3}$

Answer: $x = \frac{2}{3}$

9 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $5y = 12x + k$ is a tangent to the circle
$(x + 2)^2 + (y + 0)^2 = 4$.
Find the two possible values of k.

Substitute $y = \frac{12x + k}{5}$: $(x + 2)^2 + (\frac{12x + k}{5} + 0)^2 = 4$.

Multiply by 25 and collect: a quadratic in x with a = 169.

Tangent ⇒ discriminant = 0. This simplifies to $(k - 24)^2 = 676$.

$k - 24 = \pm 26$.

Answer: k = −2 or k = 50

10 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 24 m from O, and reaches a maximum height of 15 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take g = 9.8 m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 15$, so $u_y = 17.15$

Time of flight = $\frac{2u_y}{9.8}$ = 3.499 s, so $u_x = \frac{24}{3.499}$ = 6.859

$\tan\theta = \frac{u_y}{u_x}$ = 2.500, so θ = 68.2°

Speed = $\sqrt{(u_x^2 + u_y^2)}$ = 18.5

Answer: θ = 68.2°, speed 18.5 m s^{-1}