Day 117 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^2 - 4x + 4$ is translated by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

and then a stretch, scale factor 3, parallel to the y-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$: $y = f(x - 3) - 2 = x^2 - 10x + 23$.

Then a stretch, scale factor 3, parallel to the y-axis: multiply y by 3.

Expand and simplify to give the quadratic.

Answer: $y = 3x^2 - 30x + 69$

2 Poisson: scaling the rate, then binomial (CIE)

Emails arrive in an inbox at an average rate of 12 per hour, independently and at random.

(a) Find the probability of at least 2 in a 20-minute period.

(b) Find the probability that, of 8 separate 20-minute periods, exactly 3 have no arrivals.

20-minute period: $\lambda = 12 \times \frac{20}{60} = 4$, so $X \sim Po(4)$

(a) $P(X \geq 2) = 1 - P(X \leq 1) = 0.9084$

(b) $P(X = 0) = e^{-4} = 0.01832$

$Y \sim B(8, 0.01832)$: $P(Y = 3) = 0.000314$

Answer: (a) 0.908 (b) 0.000314

3 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2\sec^2\theta + \tan\theta - 5 = 0$

for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\sec^2\theta \equiv 1 + \tan^2\theta$: $2\tan^2\theta + \tan\theta - 3 = 0$

$(2\tan\theta + 3)(\tan\theta - 1) = 0$

$\tan\theta = -\frac{3}{2}$ or $\tan\theta = 1$

$\theta = 123.7°$ and $45.0°$ $(+180°)$

Answer: θ = 45.0°, 123.7°, 225.0°, 303.7°

4 Binomial expansion (integer powers) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the term independent of x in the expansion of $(2x^2 - \frac{2}{x^2})^8$.

General term: $C(8,r)(2x^2)^{8-r}(-\frac{2}{x^2})^r$.

Power of x: $2(8 - r) - 2r = 0$, so $r = 4$.

Term = $C(8,4) \times 2^4 \times (-2)^4 = 17920$.

Answer: 17920

5 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{x^2 + 6x + 3}{(x + 3)(x + 5)} \equiv 1 + \frac{A}{x + 3} + \frac{B}{x + 5}$.

(b) Hence find the exact value of $\int_0^1 \frac{x^2 + 6x + 3}{(x + 3)(x + 5)}\,dx$.

Numerator = $(x + 3)(x + 5) + A(x + 5) + B(x + 3)$

$x = -3$ gives $A = -3$; $x = -5$ gives $B = 1$

Integral = $[x + -3\ln(x + 3) + 1\ln(x + 5)]$

Substitute limits 0 and 1

Answer: A = −3, B = 1; 1 − 5ln 2 + 4ln 3 − ln 5

6 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + 2n - 2$, $u_1 = 4$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 222$.

$$u_n = u_1 + \sum_{k=1}^{n-1}(2k - 2) = 4 + 2 \times \frac{(n-1)n}{2} - 2(n-1).$$

$= n^2 - 3n + 6$.

So $a = 1$, $b = -3$, $c = 6$.

$u_{16} = 214 \leq 222$ and $u_{17} = 244 > 222$.

Answer: (a) $a = 1, b = -3, c = 6$ (b) $n = 17$

7 Car on a hill: power and maximum speed (CIE)

A car of mass 800 kg, with engine power 8.136 kW, climbs a straight road with $\sin\alpha = \frac{1}{25}$ against a constant resistance of 500 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at 5 m s^{-1}. Take $g = 9.8$ m s^{-2}.

Steady speed: driving force $= 500 + 800 \times 9.8 \times \frac{1}{25} = 813.6$ N

$P = F v$, so $v = \frac{8136}{813.6} = 10$

At 5 m s^{-1}: $F = \frac{8136}{5} = 1627.2$ N

$F - 813.6 = 800\,a$, so $a = \frac{813.6}{800}$

Answer: (a) 10 m s^{-1} (b) 1.02 m s^{-2}

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates $(-3, 1, 4)$, $(2, 3, -2)$, $(4, -3, 0)$.

Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$\mathrm{BA} = \begin{pmatrix} -5 \\ -2 \\ 6 \end{pmatrix}$, $\mathrm{BC} = \begin{pmatrix} 2 \\ -6 \\ 2 \end{pmatrix}$; $\mathrm{BA} \cdot \mathrm{BC} = 14$

$|\mathrm{BA}| = \sqrt{65}$, $|\mathrm{BC}| = \sqrt{44}$

$\cos \mathrm{ABC} = \frac{14}{\sqrt{65}\sqrt{44}} = 0.2618$, so $\mathrm{ABC} = 74.8°$

$|\mathrm{BA} \times \mathrm{BC}| = \left|\begin{pmatrix} 32 \\ 22 \\ 34 \end{pmatrix}\right| = \sqrt{2664}$

Area $= \frac{1}{2}|\mathrm{BA} \times \mathrm{BC}|$

Answer: ABC = 74.8°, area = $3\sqrt{74}$

9 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A game costs £1. Two fair dice are rolled: a total of at least 11 wins £14, a total of at most 3 wins £3, otherwise nothing.
(a) Find the expected profit.
(b) Find the standard deviation of the profit (3 s.f.).
(c) Find the stake that makes the game fair.

P(total ≥ 11) = $\frac{1}{12}$, P(total ≤ 3) = $\frac{1}{12}$, otherwise $\frac{5}{6}$

Profit: £13 with $\frac{1}{12}$, £2 with $\frac{1}{12}$, £-1 with $\frac{5}{6}$

(a) E = $\frac{5}{12}$

(b) E(P^2) = $\frac{61}{4}$, Var = $\frac{61}{4} - (\frac{5}{12})^2 = \frac{2171}{144}$, so $\sigma = 3.883$

(c) Fair when the stake equals E(winnings) = $\frac{1}{12} \times 14 + \frac{1}{12} \times 3 = \frac{17}{12}$

Answer: (a) £$\frac{5}{12}$ (b) 3.88 (c) £$\frac{17}{12}$

10 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector ($30\mathbf{i} + 15\mathbf{j}$) km relative to a port O, and moves with constant velocity ($-2\mathbf{i} + \mathbf{j}$) km h^{-1}. Ship B leaves O at t = 0 with constant speed 10 km h^{-1} and intercepts A.
Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: ($30\mathbf{i} + 15\mathbf{j}$) + t($-2\mathbf{i} + \mathbf{j}$). B at time t: t v_B (unknown velocity)
At interception B has travelled 10 t, so $|(30\mathbf{i} + 15\mathbf{j}) + t(-2\mathbf{i} + \mathbf{j})| = 10t$
Squaring and rearranging: $-95t^2 - 90t + 1125 = 0$, which gives t = 3 or a negative value
t = 3 (the other root is negative since B is faster than A)

A is then at ($30\mathbf{i} + 15\mathbf{j}$) + 3($-2\mathbf{i} + \mathbf{j}$) = ($24\mathbf{i} + 18\mathbf{j}$), so v_B = $\frac{1}{3}(24\mathbf{i} + 18\mathbf{j}) = (8\mathbf{i} + 6\mathbf{j})$

Answer: (a) 3 h (b) ($8\mathbf{i} + 6\mathbf{j}$) km h^{-1}