Day 116 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Regression line from summations (AQA, Edexcel, OCR, MEI)

For n = 12 pairs of values: $\Sigma x = 120$, $\Sigma y = 132$, $\Sigma x^2 = 1337$, $\Sigma xy = 1731$.
(a) Find the equation of the regression line of y on x.
(b) The x values lie between 6 and 15. Estimate y when x = 15 and comment on its reliability.

$S_{xx} = 1337 - \frac{120^2}{12} = 137$

$S_{xy} = 1731 - \frac{120 \times 132}{12} = 411$

$b = \frac{S_{xy}}{S_{xx}} = \frac{411}{137} = 3$

$a = \bar{y} - b\,\bar{x} = 11 - (3)(10) = -19$

$y = -19 + 3x$

At x = 15: y = 26. This is extrapolation (outside the data range), so may be unreliable

Answer: (a) $y = -19 + 3x$ (b) 26, unreliable (extrapolation)

2 Rough table and smooth slope: string cut (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particle A (4 kg) lies on a rough table with $\mu = 0.1$, and is joined by a light string over a smooth pulley (see diagram) to particle B (8 kg) on a smooth slope, $\alpha = 30°$.
Released from rest, the string is cut when B has moved 3.75 m. Find the further distance A slides before stopping.

A: $T - 0.1 \times 4 \times 9.8 = 4a$. B: $8 \times 9.8 \times \frac{1}{2} - T = 8a$

Adding: $9.8(8 \times \frac{1}{2} - 0.1 \times 4) = 12a$, so $a = 2.94$

$T = 4(2.94 + 0.1 \times 9.8) = 15.68$

When cut: $v^2 = 2 \times 2.94 \times 3.75 = 22.05$. Then A decelerates at $\mu g = 0.98$

Further distance = $\frac{22.05}{2 \times 0.98} = 11.25$

Answer: 11.25 m

3 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{7 + 2\sqrt{2}}{3 - 3\sqrt{2}}$ in the form $p + q\sqrt{2}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(3 + 3\sqrt{2})$.
Numerator: $(7 + 2\sqrt{2})(3 + 3\sqrt{2}) = 33 + 27\sqrt{2}$.
Denominator: $3^2 - 3^2(2) = -9$.
So the fraction = $\frac{33 + 27\sqrt{2}}{-9}$.

Answer: $-\frac{11}{3} - 3\sqrt{2}$

4 Particle on a rough table, string slack when B lands (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (4 kg) and B (3 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a rough table with $\mu = 0.5$, and B is 2.8 m above the floor.
The system is released from rest. Find the total distance A moves before coming to rest (A does not reach the pulley).

Until B lands, A: $T - 0.5 \times 4 \times 9.8 = 4a$. B: $3 \times 9.8 - T = 3a$
Adding: $(3 - 0.5 \times 4) \times 9.8 = 7a$, so $a = 1.4$
$v^2 = 2 \times 1.4 \times 2.8 = 7.84$, so $v = 2.8$

After B lands, A decelerates at $\mu g = 4.9$: extra distance = $\frac{2.8^2}{2 \times 4.9} = 0.8$

Total distance = 2.8 + 0.8
Answer: 3.6 m

5 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution u = cos x to find the exact value of

$\int_0^{\pi/3} \sin x \cos^4 x \, dx$

du = −sin x dx; limits become u = 1 and u = cos(π/3)

Integral = $\int_{\cos(\pi/3)}^{1} u^4 \, du$

Evaluate the limits

Answer: $\frac{31}{160}$

6 Proof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Sam claims that $n^2 - n + 41$ is prime for every positive integer n.
Show that Sam is wrong by finding the smallest value of n that is a counter-example.

Test n = 1, 2, 3, ... in turn until the value is not prime.

For n = 41: $41^2 - 41 + 41 = 1681$.

1681 = 41 × 41, which is not prime.

All smaller values of n give a prime, so n = 41 is the smallest.

Answer: n = 41 (value 1681 = 41 × 41)

7 Normal then binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The mass of a apple is $N(56, 7^2)$ g. A apple is 'heavy' if it exceeds 63.
(a) Find the probability that a apple is heavy.
(b) 11 apples are chosen at random. Find the probability that at least 4 are heavy.

(a) P(X > 63) = P(Z > 1.000) = 0.1587

(b) Let Y = number heavy, Y ~ B(11, 0.1587)

P(Y ≥ 4) = 1 − P(Y ≤ 3) = 0.0824

Answer: (a) 0.159 (b) 0.0824

8 Area between a curve and a line (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curves $y = e^x$ and $y = 81\,e^{-x}$ meet at the point P.
Find the exact area of the region bounded by the two curves
and the y-axis.

$e^x = 81\,e^{-x}$ gives $e^{2x} = 81$, so $e^x = 9$, x = ln 9

Area = $\int_0^{\ln 9} (81\,e^{-x} - e^x)\,dx = [-81\,e^{-x} - e^x]$

= (−9 − 9) − (−81 − 1)

Answer: 64

9 Conditions on the roots of a quadratic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

One root of $x^2 + kx - 27 = 0$ is the square of the other root. Find the possible values of k.

Let the roots be α and α^2.

Product: $\alpha^3 = -27$, so α = −3 (the only real cube root).

Sum: $\alpha + \alpha^2 = -3 + 9 = 6 = -k$.

Answer: k = −6

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -5 \\ 4 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$.

The point P has coordinates $(-5, -1, 7)$.

Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathbf{FP} = \begin{pmatrix} 0 \\ -5 \\ 3 \end{pmatrix} - t\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

$\mathbf{FP} \cdot \mathbf{d} = 0$: $5 - t(5) = 0$, so $t = 1$

F = $(-3, 3, 4)$

$\mathbf{PF} = \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}$, so PF = $\sqrt{29}$

Answer: F = $(-3, 3, 4)$, PF = $\sqrt{29}$