Day 115 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Time above a given height (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is thrown vertically upwards from ground level. It is more than 191.1 m above the ground for 10 s.
Find the speed of projection and the greatest height reached. Take g = 9.8 m s^{-2}.

By symmetry, the ball takes 5 s to rise from height 191.1 m to its highest point
Speed at height 191.1 m: $v = 9.8 \times 5 = 49$ m s^{-1}
Speed of projection: $u^2 = v^2 + 2gh = 2401 + 2 \times 9.8 \times 191.1 = 6146.56$
$u = 78.4$. Greatest height = $\frac{u^2}{(2g)} = \frac{6146.56}{19.6} = 313.6$

Answer: 78.4 m s^{-1}, 313.6 m

2 Modelling y = ab^x and y = ax^n (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = ab^x$ passes through (0, 4) and (3, 256).
Find the exact values of a and b, and the value of y when x = 4.

$ab^0 = 4$ and $ab^3 = 256$
Divide: $b^3 = 64$ so $b = 4$
$a = 4 \div 4^0 = 4$
$y = 4 \times 4^4$
Answer: a = 4, b = 4, y = 1024

3 Force up a rough slope: find weight and friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle of weight W N is on a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{12}{13}$. A force P N acts up a line of greatest slope; the particle stays at rest for $20 \leq P \leq 28$.
Find (a) W, (b) the coefficient of friction, (c) the acceleration when P = 56. Take g = 9.8 m s^{-2}.

Limiting friction acts down the plane at P = 28 and up the plane at P = 20
$P_{max} = W\sin\alpha + F$ and $P_{min} = W\sin\alpha - F$, where $F = \mu W\cos\alpha$
Adding: $2W\sin\alpha = 48$, so $W = \frac{48}{2 \times \frac{12}{13}} = 26$
Subtracting: $2F = 8$, so $F = 4 = \mu \times 26 \times \frac{5}{13}$, giving $\mu = 0.4$
P = 56: net force up the plane = 56 − 28 = 28, and mass = $\frac{26}{9.8}$

Answer: (a) 26 N (b) 0.4 (c) 10.6 m s^{-2}

4 Composite functions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = -2x + 3$ and $g(x) = x^2 - 2$.
Solve fg(x) = gf(x).

$fg(x) = -2(x^2 - 2) + 3 = -2x^2 + 7$.
$gf(x) = (-2x + 3)^2 - 2 = 4x^2 - 12x + 7$.
Equate: $6x^2 - 12x = 0$.
Solve the quadratic: x = 0 or x = 2.
Answer: x = 0 or x = 2

5 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is acute. $\sin A = \frac{5}{13}$ and $\sin B = \frac{7}{25}$.

Find the exact values of $\cos(A + B)$ and $\sin(A - B)$.

$\cos A = \frac{12}{13}$, $\sin A = \frac{5}{13}$; $\sin B = \frac{7}{25}$, $\cos B = \frac{24}{25}$

$\cos(A + B) = \cos A \cos B - \sin A \sin B = \frac{253}{325}$

$\sin(A - B) = \sin A \cos B - \cos A \sin B = \frac{36}{325}$

Answer: $\cos(A + B) = \frac{253}{325}$, $\sin(A - B) = \frac{36}{325}$

6 Self-inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{kx - 3}{x - 5}$ for $x \neq 5$. Given that f is self-inverse, find k, and hence solve $f(x) = x$.

Let $y = f(x)$: $y(x - 5) = kx - 3$, so $x(y - k) = 5y - 3$.

$f^{-1}(x) = \frac{5x - 3}{x - k}$.

$f = f^{-1}$ requires $k = 5$.

$f(x) = x$: $5x - 3 = x(x - 5)$, so $x^2 - 10x + 3 = 0$.

$x = 5 \pm \sqrt{22} = 5 \pm \sqrt{22}$.

Answer: $k = 5$; $x = 5 \pm \sqrt{22}$

7 Cumulative distribution function (CIE)

The cdf of X is $F(x) = 0$ for $x < 0$, $F(x) = \frac{x^4}{16}$ for $0 \leq x \leq 2$, and $F(x) = 1$ for $x > 2$.

(a) Find the pdf of X.
(b) Find E(X).
(c) Find $P(0.5 < X < 1)$ and the median, to 3 significant figures.

(a) $f(x) = F'(x) = \frac{1}{4} x^3$ for $0 \leq x \leq 2$

(b) $E(X) = \int x f(x)\, dx$ from 0 to 2 $= \frac{8}{5}$

(c) $P = F(1) - F(0.5) = 0.05859$

Median m: $F(m) = 1/2$, so $m = 2 \times (1/2)^{1/4}$

$m = 1.682$

Answer: (a) $f(x) = \frac{1}{4} x^3$ (b) $\frac{8}{5}$ (c) 0.0586, median 1.68

8 Alternate turns and sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A and B take turns, A first, to score; the first to score wins. P(A scores) $= \frac{1}{5}$, P(B scores) $= \frac{1}{2}$ each attempt.

Find P(A wins (a) on attempt 1 or 2, (b) overall).

P(A wins at attempt 1) $= \frac{1}{5}$; to reach attempt 2 both miss first: $(\frac{4}{5})(\frac{1}{2}) = \frac{2}{5}$

(a) $\frac{1}{5} + \frac{2}{5} \times \frac{1}{5} = \frac{7}{25}$

(b) P(A wins) $= \frac{1}{5} + \frac{2}{5} \times \frac{1}{5} + \frac{2}{5}^2 \times \frac{1}{5} + \ldots$

Geometric: first term $\frac{1}{5}$, common ratio $\frac{2}{5}$

$S = \frac{a}{1 - r} = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}$

Answer: (a) $\frac{7}{25}$ (b) $\frac{1}{3}$

9 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{2x - 8}{x - 3}$.

(a) Write down the equations of the asymptotes of C.
(b) The line $y = 2x + k$ is a tangent to C.
Find the possible values of k.

(a) Vertical: $x = 3$. As $x \to \infty$, $y \to 2$, so $y = 2$.
(b) $2x - 8 = (2x + k)(x - 3)$.
$2x^2 + (k - 8)x + (-3k + 8) = 0$.
Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 + 8k = 0$.
Answer: (a) $x = 3$, $y = 2$ (b) $k = -8$ or $k = 0$

10 Perpendicular from a point to a line (3D) (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} -4 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}$.

The point P has coordinates $(-2, -2, 0)$.
Find the coordinates of the foot F of the perpendicular from P to l, and the exact distance PF.

F = (a + td) for some t, so $\mathrm{FP} = \begin{pmatrix} 2 \\ -4 \\ 3 \end{pmatrix} - t\begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix}$
$\mathrm{FP} \cdot \mathbf{d} = 0$: $-27 - t(27) = 0$, so $t = -1$
F = $(-1, -1, 0)$
$\mathrm{PF} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$, so $\mathrm{PF} = \sqrt{2}$
Answer: F = $(-1, -1, 0)$, PF = $\sqrt{2}$