Day 113 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Arithmetic series modelling (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The first row of a theatre has 22 seats. Each row has 4 more seats than the row in front. Find
(a) the number of seats in row 36,
(b) the least number of rows for at least 604 seats.

Arithmetic: a = 22, d = 4.
(a) $u_{36} = 22 + 35 \times 4 = 162$.
(b) $S_n = \frac{n}{2}(44 + 4n - 4) = \frac{n(4n + 40)}{2} \geq 604$.
$S_{13} = 598 < 604$, $S_{14} = 672 \geq 604$.
Answer: (a) 162 (b) 14 rows

2 Binomial: finding p from a ratio (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$X \sim B(10, p)$. It is given that $P(X = 2) = \frac{9}{8} \times P(X = 1)$.
(a) Find the value of p.
(b) Find P(X = 0), correct to 3 significant figures.

$\frac{90}{2} p^2(1 - p)^8 = \frac{9}{8} \times 10\, p(1 - p)^9$
Divide by $p(1 - p)^8$: $4.5\, p = \frac{9}{8}(1 - p)$
$(4.5 + \frac{9}{8})p = \frac{9}{8}$
$p = \frac{1}{5}$
$P(X = 0) = (1 - p)^{10} = (\frac{4}{5})^{10} = 0.107$
Answer: (a) p = $\frac{1}{5}$ (b) 0.107

3 Integration by substitution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use the substitution u = 1 + ln x to find the exact value of
$\int_1^{e^3} \frac{1}{x(1 + \ln x)}\, dx$
$du = \frac{1}{x}\, dx$; limits x = 1, e^3 become u = 1, 4
Integral = $\int_1^4 \frac{1}{u}\, du = [\ln u]$
Evaluate the limits
Answer: 2ln 2

4 Periodic recurrence relations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sequence is defined by $u_1 = 4$ and
$u_{n+1} = \frac{1}{1 - u_n}$.
Show that the sequence is periodic and find
$\sum_{r=1}^{60} u_r$.

$u_1 = 4$, $u_2 = -\frac{1}{3}$, $u_3 = \frac{3}{4}$, $u_4 = 4$.
$u_4 = u_1$, so the sequence repeats with period 3.
One period sums to $\frac{53}{12}$.
60 = 3 × 20 + 0: 20 full periods plus the first 0 terms.
Answer: $\frac{265}{3}$

5 Direction of motion at 45 degrees (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from level ground with horizontal and vertical velocity components 34.3 and 53.9 m s^{-1}.
Find (a) the two times at which it moves at 45° to the horizontal, (b) the horizontal distance between those two points.
Take g = 9.8 m s^{-2}.

At 45° to the horizontal, $|v_y| = v_x = 34.3$
Moving upwards: $53.9 - 9.8t = 34.3$, so $t = 2$
Moving downwards: $53.9 - 9.8t = -34.3$, so $t = 9$
Horizontal distance = $34.3 \times (9 - 2)$
Answer: (a) 2 s and 9 s (b) 240.1 m

6 Exponential and power models by logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Data satisfy $y = ab^x$. The line of best fit for lg y against x is $\lg y = 0.7 + 0.5x$.
(a) Find a and b to 3 significant figures.
(b) Estimate y when x = 17, to 3 significant figures.
(c) Interpret the value of b.

$y = ab^x$ gives $\lg y = \lg a + x \lg b$
$\lg a = 0.7$ so $a = 10^{0.7} = 5.012$
$\lg b = 0.5$ so $b = 10^{0.5} = 3.162$
$x = 17$: $\lg y = 0.7 + 0.5 \times 17 = 9.2$
$y = 10^{9.2} = 1580000000$
b is the factor y is multiplied by each time x increases by 1 (growth of 220%)
Answer: (a) a = 5.01, b = 3.16 (b) 1580000000 (c) y grows by a factor of 3.16 per unit of x

7 Line tangent to a hyperbola (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $y = \frac{3x + 18}{x + 3}$.

(a) Write down the equations of the asymptotes of C.
(b) The line $y = -x + k$ is a tangent to C.
Find the possible values of k.

(a) Vertical: $x = -3$. As $x \to \infty$, $y \to 3$, so $y = 3$.
(b) $3x + 18 = (-x + k)(x + 3)$.
$-x^2 + (k - 6)x + (3k - 18) = 0$.
Tangent: $b^2 - 4ac = 0$, which simplifies to $k^2 - 36 = 0$.
Answer: (a) $x = -3$, $y = 3$ (b) $k = -6$ or $k = 6$

8 Tangent and axes triangle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tangent to $y = \frac{24}{x^2 + 3}$ at the point where x = 3
meets the axes at A and B. Find the exact area of triangle OAB,
where O is the origin.

At x = 3: y = 2. Differentiate: gradient = −1
Tangent: $y - 2 = (-1)(x - 3)$, so $y = -x + 5$
y-intercept 5; x-intercept 5

Area = $\frac{1}{2} \times |5| \times |5|$

Answer: $\frac{25}{2}$

9 Variable force in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P of mass 2 kg moves under a single force F N, where $\mathbf{F} = (12t + 10)\mathbf{i} + (18t^2 - 10)\mathbf{j}$ and t is the time in seconds. At t = 0, P is at the origin with velocity $(3\mathbf{i} + 2\mathbf{j})$ m s^{-1}.
Find (a) the velocity of P at t = 2, (b) the position vector of P at t = 2.

$\mathbf{a} = \frac{\mathbf{F}}{2} = (6t + 5)\mathbf{i} + (9t^2 - 5)\mathbf{j}$

$\mathbf{v} = \int \mathbf{a}\, dt = (3t^2 + 5t + 3)\mathbf{i} + (3t^3 - 5t + 2)\mathbf{j}$ (using $\mathbf{v}(0) = (3\mathbf{i} + 2\mathbf{j})$)
At t = 2: $\mathbf{v} = (25\mathbf{i} + 16\mathbf{j})$
$\mathbf{r} = \int \mathbf{v}\, dt = (t^3 + 2.5t^2 + 3t)\mathbf{i} + (0.75t^4 - 2.5t^2 + 2t)\mathbf{j}$
At t = 2: $\mathbf{r} = (24\mathbf{i} + 6\mathbf{j})$
Answer: (a) $(25\mathbf{i} + 16\mathbf{j})$ m s^{-1} (b) $(24\mathbf{i} + 6\mathbf{j})$ m

10 Separable differential equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$\frac{dy}{dx} = \frac{3x^2}{y}$, y > 0, and y = 3 when x = 0.
Find y in terms of x, and the value of y when x = 2.

$\int y\, dy = \int 3x^2\, dx$ gives $\frac{y^2}{2} = 1x^3 + c$

x = 0, y = 3: $c = \frac{9}{2}$, so $y^2 = 2x^3 + 9$

x = 2: $y^2 = 25$

Answer: $y = \sqrt{2x^3 + 9}$; y = 5