Day 112 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Partial fractions (improper) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{x^3 - 3x^2 - 2x + 14}{(x + 1)(x - 3)}$ in the form $ax + b + \frac{C}{(x + 1)} + \frac{D}{(x - 3)}$.

Divide: the numerator has degree 3, the denominator degree 2.
$(x^3 - 3x^2 - 2x + 14) \div (x^2 - 2x - 3)$ gives quotient $x - 1$.
Remainder: $-x + 11$.
Put $x = -1$: remainder $= C(-4)$ so $C = -3$. Put $x = 3$: $D = 2$.

Answer: $x - 1 - \frac{3}{(x + 1)} + \frac{2}{(x - 3)}$

2 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass M kg, carrying a passenger of mass 60 kg, hangs from a vertical cable. The tension in the cable is 3224 N and the floor exerts a force of 624 N on the passenger.
Find (a) the acceleration of the lift, with its direction, (b) the value of M. Take $g = 9.8$ m s^{-2}.

Passenger (upwards positive): $624 - 60 \times 9.8 = 60\,a$, so $a = 0.6$
$a = 0.6$ m s^{-2}, so the acceleration is upwards
Lift and passenger: $3224 - (M + 60) \times 9.8 = (M + 60) \times 0.6$
$(M + 60)(9.8 + 0.6) = 3224$, so $M + 60 = \frac{3224}{10.4} = 310$

Answer: (a) 0.6 m s^{-2} upwards (b) M = 250

3 Coded data: mean, standard deviation, Σx^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

For 20 observations x, $\Sigma(x - 30) = 40$ and $\Sigma(x - 30)^2 = 680$.
(a) Find the mean of x.
(b) Find the standard deviation of x.
(c) Find the value of Σx^2.

Mean of $(x - 30) = \frac{40}{20} = 2$, so mean of $x = 32$

Variance $= \frac{680}{20} - (2)^2 = 30$

$\sigma = \sqrt{30}$
$\Sigma x^2 = \Sigma(x - 30)^2 + 60\,\Sigma(x - 30) + 20 \times 30^2$
$= 680 + 2400 + 18000$
Answer: (a) 32 (b) 5.48 (c) 21080

4 Trig equations with compound angles (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\tan(2\theta + 40°) = \frac{1}{\sqrt{3}}$
for $0° \le \theta \le 360°$.

Let $u = 2\theta + 40°$; then $\tan u = \frac{1}{\sqrt{3}}$
Base solutions: $u = 30°$ (+ multiples of 180°)
$\theta = (u - 40°) \div 2$, keeping those in range
Answer: θ = 85°, 175°, 265°, 355°

5 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 32 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan\alpha = \frac{4}{3}$ and $\tan\beta = \frac{3}{4}$.

Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 41.6 N.

Horizontally: $T_1 \cos\alpha = T_2 \cos\beta$, so $T_1 \times \frac{3}{5} = T_2 \times \frac{4}{5}$

Vertically: $T_1 \sin\alpha + T_2 \sin\beta = 32$, and $T_2 = \frac{3}{4} T_1$

$T_1(\frac{4}{5} + \frac{3}{4} \times \frac{3}{5}) = 32$, so $T_1 = 25.6$ and $T_2 = 19.2$

The tension in AP is the larger: it is 25.6 when W = 32

Tensions are proportional to W, so $W_{max} = 32 \times \frac{41.6}{25.6} = 52$

Answer: (a) AP: 25.6 N, BP: 19.2 N (b) 52 N

6 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = e^x + 5x - 7$.

(a) Show that $f(x) = 0$ has a root α with $0 < \alpha < 1$.

(b) Use Newton-Raphson with $x_0 = 0$ to find x_1 and x_2 to 4 decimal places.

$f(0) = -6.00$ and $f(1) = 0.718$: a change of sign, f continuous

$f'(x) = e^x + 5$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 0 - \frac{-6.000}{6.000} = 1.0000$

$x_2 = 1.0000 - \frac{0.7183}{7.718} = 0.9069$

Answer: $x_1 = 1.0000$, $x_2 = 0.9069$

7 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + n - 4$, $u_1 = -5$.

(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.

(b) Find the least n for which $u_n > 417$.

$u_n = u_1 + \sum_{k=1}^{n-1}(k - 4) = -5 + 1 \times \frac{(n-1)n}{2} - 4(n-1)$.

$= \frac{1}{2}n^2 - \frac{9}{2}n - 1$.

So $a = \frac{1}{2}$, $b = -\frac{9}{2}$, $c = -1$.

$u_{33} = 395 \leq 417$ and $u_{34} = 424 > 417$.

Answer: (a) $a = \frac{1}{2}$, $b = -\frac{9}{2}$, $c = -1$ (b) $n = 34$

8 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

An open-topped box has a square base of side x m and volume 32 m^3.

Show that the surface area is $A = x^2 + \frac{128}{x}$, and find the minimum value of A, justifying that it is a minimum.

$h = \frac{32}{x^2}$, $A = x^2 + 4xh = x^2 + \frac{128}{x}$

$dA/dx = 2x - \frac{128}{x^2} = 0$, so $x^3 = 64$, $x = 4$

$d^2A/dx^2 = 2 + \frac{256}{x^3} > 0$, so a minimum

$A = 16 + \frac{128}{4}$

Answer: 48 m^2

9 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $8y = 6x + k$ is a tangent to the circle $(x + 2)^2 + (y + 1)^2 = 1$.

Find the two possible values of k.

Substitute $y = \frac{6x + k}{8}$: $(x + 2)^2 + (\frac{6x + k}{8} + 1)^2 = 1$.

Multiply by 64 and collect: a quadratic in x with a = 100.

Tangent ⇒ discriminant = 0. This simplifies to $(k - 4)^2 = 100$.

$k - 4 = \pm 10$.

Answer: $k = -6$ or $k = 14$

10 Critical region, significance level and Type II error (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A test of H_0: $p = 0.2$ against H_1: $p < 0.2$ uses a sample of 50 at the 5% level.

(a) Find the critical region for X, the number of successes.

(b) Find the actual significance level.

(c) Find P(Type II error) when in fact $p = 0.1$.

$X \sim B(50, 0.2)$. $P(X \leq 5) = 0.0480$ and $P(X \leq 6) = 0.103$

The first is at most 0.05 and the second is not, so the critical region is $X \leq 5$

Actual significance level = $P(X \leq 5) = 0.0480$

Type II error: H_0 is not rejected, so $X \geq 6$, with $X \sim B(50, 0.1)$

$P(X \geq 6) = 1 - P(X \leq 5) = 0.384$

Answer: (a) $X \leq 5$ (b) 0.0480 (c) 0.384