Day 111 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 6 m. After each bounce it rises to $\frac{1}{2}$ of its previous height.

(a) Find the total distance it travels before it stops.
(b) After how many bounces does it rise less than 0.01 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$= H + \frac{2Hr}{1 - r} = 6 + \frac{2 \times 6 \times \frac{1}{2}}{1 - \frac{1}{2}} = 18.$

(b) Rebound height after n bounces $= 6(\frac{1}{2})^n < 0.01.$

$n > \frac{\log(0.00166667)}{\log(0.5)} = 9.23$, so $n = 10.$

Answer: (a) 18 m (b) 10 bounces

2 Discrete random variable with a probability formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

P(X = x) = k(x + 4) for x = 1, 2, 3, and 0 otherwise.

(a) Show that $k = \frac{1}{18}$.
(b) Find E(X) and Var(X).
(c) Find Var(2X + 2).

(a) Σ P(X = x) = k(5 + 6 + 7) = 18 k = 1, so $k = \frac{1}{18}$

(b) $E(X) = \Sigma$ x P(X = x) $= \frac{1\times5 + 2\times6 + 3\times7}{18} = \frac{19}{9}$

$E(X^2) = \frac{1\times5 + 4\times6 + 9\times7}{18} = \frac{46}{9}$

$Var(X) = \frac{46}{9} - \frac{19}{9}^2 = \frac{53}{81}$

(c) $Var(2X + 2) = 2^2 Var(X) = 4 \times \frac{53}{81} = \frac{212}{81}$

Answer: (b) E(X) = $\frac{19}{9}$, Var(X) = $\frac{53}{81}$ (c) $\frac{212}{81}$

3 Resultant of forces: unknown constant and acceleration (AQA, Edexcel, OCR, MEI, Eduqas)

Three forces $(-2\mathbf{i} - 2\mathbf{j})$ N, $(-3\mathbf{i} + k\mathbf{j})$ N and $(-5\mathbf{i} - 5\mathbf{j})$ N act on a particle of mass 4 kg. The resultant is parallel to $(-5\mathbf{i} - \mathbf{j})$.
Find (a) the value of k, (b) the magnitude of the acceleration of the particle, to 3 s.f.

Resultant $= (-10)\mathbf{i} + (-7 + k)\mathbf{j}$

Parallel to $(-5\mathbf{i} - \mathbf{j})$: $\frac{-7 + k}{-10} = \frac{-1}{-5}$, so $k = 5$

$\mathbf{R} = (-10\mathbf{i} - 2\mathbf{j})$ N, so $|R| = \sqrt{(-10^2 + -2^2)} = 10.20$
$a = |R| \div 4$

Answer: (a) k = 5 (b) 2.55 m s^{-2}

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4 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = 2x^2 - 12x + 24$
for $x \geq k$. Given that k is the least value for which
f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = 2(x - 3)^2 + 6$.
The vertex is at x = 3, so k = 3 (f is one-one on one side of it).

Let y = f(x): $y - 6 = 2(x - 3)^2$, so $(x - 3) = \pm\sqrt{\frac{y - 6}{2}}$.

As $x \geq 3$, take the positive root: $x = 3 + \sqrt{\frac{y - 6}{2}}$.

Range of f = domain of f^{-1}: f(3) = 6, so $x \geq 6$.

Answer: $k = 3$; $f^{-1}(x) = 3 + \sqrt{\frac{x - 6}{2}}$; $x \geq 6$

5 Integrals of the form f'(x)/f(x) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact value of

$$\int_0^1 \frac{x}{x^2 + 3}\,dx$$

Numerator is $\frac{1}{2}$ × derivative of the denominator

Integral = $[\frac{1}{2}\ln(x^2 + 3)]$

$= \frac{1}{2}(\ln 4 - \ln 3)$

Answer: $\ln 2 - \frac{1}{2}\ln 3$

6 Tangent to a circle and triangle area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The circle $x^2 + y^2 + 2x - 6y - 90 = 0$
passes through P(−7, −5). The tangent at P meets the axes at A and B. Find the area of triangle OAB.

Centre (−1, 3). Gradient of radius CP = $\frac{4}{3}$.

Tangent gradient = $-\frac{3}{4}$: $-6x - 8y = 82$.

x = 0: $y = -\frac{41}{4}$. y = 0: $x = -\frac{41}{3}$.

Area = ½ × $\frac{41}{3}$ × $\frac{41}{4}$.

Answer: $\frac{1681}{24}$

7 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\sin 3\theta \equiv 3\sin\theta - 4\sin^3\theta$, solve
$\sin 3\theta = -\sin\theta$ for $0 < \theta < 2\pi$, in terms of π.

Substitute: $\sin\theta(3 - 4\sin^2\theta + 1) = 0$
$\sin\theta = 0$ or $\sin^2\theta = 1$
$\sin\theta = 0$ gives π

$\sin^2\theta = 1$ gives $\frac{\pi}{2}$, $\frac{3\pi}{2}$

Answer: $\theta = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$

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8 Car on a hill: power and maximum speed (CIE)

A car of mass 1500 kg, with engine power 9.88 kW, climbs a straight road with $\sin\alpha = \frac{1}{25}$ against a constant resistance of 400 N.

Find (a) the greatest steady speed up the hill, (b) the acceleration at 5 m s^{-1}. Take g = 9.8 m s^{-2}.

Steady speed: driving force = $400 + 1500 \times 9.8 \times \frac{1}{25} = 988$ N

$P = F v$, so $v = \frac{9880}{988} = 10$

At 5 m s^{-1}: $F = \frac{9880}{5} = 1976$ N

$F - 988 = 1500\,a$, so $a = \frac{988}{1500}$

Answer: (a) 10 m s^{-1} (b) 0.659 m s^{-2}

9 Poisson: sum of two independent variables, conditional (CIE)

X and Y are independent, X ~ Po(6) and Y ~ Po(2).

(a) State the distribution of X + Y.

(b) Given that X + Y = 5, find the probability that X = 3.

$X + Y \sim Po(8)$

$P(X = 3 \mid X + Y = 5) = \frac{P(X = 3)P(Y = 2)}{P(X + Y = 5)}$

$= \frac{e^{-6}6^3}{3! \times e^{-2}2^2/2!} \div \frac{e^{-8}8^5}{5!}$

$= C(5, 3)\,(\frac{3}{4})^3\,(\frac{1}{4})^2$, a binomial probability

$= 0.264$

Answer: (a) Po(8) (b) 0.264

10 Points of inflection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the coordinates of the points of inflection of

$y = \frac{3}{x^2 + 12}$.

$dy/dx = -\frac{6x}{(x^2 + 12)^2}$

$d^2y/dx^2 = \frac{6(3x^2 - 12)}{(x^2 + 12)^3}$

Zero when $x^2 = 4$, so $x = \pm 2$, and the sign changes either side

$y = \frac{3}{16} = \frac{3}{16}$

Answer: $(2, \frac{3}{16})$ and $(-2, \frac{3}{16})$