Day 110 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Sectors: perimeter and area (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A sector of a circle has radius r cm and angle θ radians, where $\theta < 2\pi$.
Its perimeter is 42 cm and its area is 110 cm^2.
Find the two possible pairs of values of r and θ.

Arc = $r\theta$, so $2r + r\theta = 42$ and $\frac{1}{2}r^2\theta = 110$

$r\theta = 42 - 2r$, so $\frac{1}{2}r(42 - 2r) = 110$

$r^2 - 21r + 110 = 0$, so $(r - 10)(r - 11) = 0$

$r = 10$: $r\theta = 22$, $\theta = \frac{11}{5}$ $r = 11$: $r\theta = 20$, $\theta = \frac{20}{11}$

Answer: r = 10, $\theta = \frac{11}{5}$ or r = 11, $\theta = \frac{20}{11}$

2 Combined mean and standard deviation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Class S has 20 students with mean mark 39 and standard deviation 6. Class P has 25 students with mean mark 53 and standard deviation 3.
(a) Find the mean mark of all 45 students.
(b) Find the standard deviation of all 45 students.

Σx for S = 20 × 39 = 780, Σx for P = 25 × 53 = 1325

Combined mean = $\frac{2105}{45} = 46.8$

$\Sigma x^2 = n(\sigma^2 + \text{mean}^2)$: $20(36 + 1521) + 25(9 + 2809) = 101590$

Variance = $\frac{101590}{45} - 46.778^2 = 69.395$, so $\sigma = 8.330$

Answer: (a) 46.8 (b) 8.33

3 Transformations of y = x^2 (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Describe a sequence of two transformations that maps the curve $y = x^2$ onto the curve $y = -2x^2 - 4x$.

Complete the square: $y = -2(x + 1)^2 + 2$.
Step 1: $y = x^2 \rightarrow y = -2x^2$: a stretch, scale factor 2, parallel to the y-axis and a reflection in the x-axis.
Step 2: replace x by (x + 1) and add 2: translation by $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Answer: a stretch, scale factor 2, parallel to the y-axis and a reflection in the x-axis, then translation $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

4 sec, cosec and tan/cot quadratics (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2\text{cosec}^2\theta + 5\cot\theta + 1 = 0$
for $0 \leq \theta < 360°$, giving answers to 1 d.p.

Use $\text{cosec}^2\theta \equiv 1 + \cot^2\theta$: $2\cot^2\theta + 5\cot\theta + 3 = 0$
$(2\cot\theta + 3)(\cot\theta + 1) = 0$

$\cot\theta = -\frac{3}{2}$ or $\cot\theta = -1$

$\tan\theta = -\frac{2}{3}$ or -1, so $\theta = 146.3°$ and $135.0°$ $(+180°)$

Answer: θ = 135.0°, 146.3°, 315.0°, 326.3°

5 Range and greatest height to speed and angle (AQA, Edexcel, OCR, MEI, Eduqas)

A golf ball is hit from a point O on horizontal ground. It lands 40 m from O, and reaches a maximum height of 20 m above the ground.

Find the angle of projection and the speed of projection, to 3 s.f. Take g = 9.8 m s^{-2}.

Vertical: $0 = u_y^2 - 2 \times 9.8 \times 20$, so $u_y = 19.80$

Time of flight = $\frac{2u_y}{9.8} = 4.041$ s, so $u_x = \frac{40}{4.041} = 9.899$

$\tan\theta = \frac{u_y}{u_x} = 2.000$, so $\theta = 63.4°$

Speed = $\sqrt{(u_x^2 + u_y^2)} = 22.1$

Answer: $\theta = 63.4°$, speed 22.1 m s^{-1}

6 Integration by parts (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Use integration by parts to find the exact value of

$\int_0^{\pi/2} e^x \sin x \, dx$

Let I be the integral. Apply parts twice (differentiating the trig function each time)
This returns a multiple of I, so (1 + 1)I = boundary terms

$I = \frac{e^x(\sin x - \cos x)}{2}$

Answer: $\frac{1}{2} + \frac{1}{2}e^{\frac{\pi}{2}}$

7 Range and domain by completing the square (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$g(x) = \sqrt{-x^2 + 16}$.

Find the largest possible domain of g, and the corresponding range.

Need $-x^2 + 16 \geq 0$, i.e. $(x + 4)(x - 4) \leq 0$.
Domain: $-4 \leq x \leq 4$.
Complete the square: $-x^2 + 16 = 16 - x^2$.
Greatest value of $g = \sqrt{16} = 4$, and least value is 0.
Answer: $-4 \leq x \leq 4$; $0 \leq g(x) \leq 4$

8 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:
$x - y = 5$
$3x^2 - 2xy + y^2 = 33$

From the first equation $x = 5 + y$.
Substitute into the second equation and collect terms:
this gives $2y^2 + 20y + 42 = 0$.
$y = -7$ or $y = -3$, giving $x = -2$ or $x = 2$.
Answer: (x, y) = (−2, −7) and (2, −3)

9 Speed decaying exponentially (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P moves on a straight line. The velocity of P is v m s^{-1} and its acceleration is $-0.2v^2$ m s^{-2}. At t = 0, v = 10.
Find (a) the time taken for the speed of P to halve, (b) the distance P travels in this time, to 3 s.f.

$\frac{dv}{dt} = -0.2v^2$: $\int \frac{dv}{v^2} = -0.2\int dt$, so $\frac{1}{v} = 0.2t + c$, and $\frac{1}{v} = 0.2t + \frac{1}{10}$

$v = 5$: $\frac{1}{v} = 0.2$, so $0.2t = 0.2 - 0.1$, giving $t = 0.5$

$s = \int \frac{1}{(0.2t + 1/10)} \, dt = \frac{1}{0.2} \ln(1 + 0.2 \times 10t)$

At t = 0.5: $s = \frac{1}{0.2} \ln 2$

Answer: (a) 0.5 s (b) 3.47 m

10 Distribution of the larger of two scores (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Two fair 8-sided dice, numbered 1 to 8, are rolled. M is the smaller of the two scores.

(a) Show that $P(M = m) = \frac{(17 - 2m)}{64}$ for m = 1, 2, ..., 8.

(b) Find P(M = 4) and E(M).

(a) $M \geq m$ means both scores $\geq m$: $P(M \geq m) = (\frac{8 - m + 1}{8})^2$, so $P(M = m) = \frac{(8 - m + 1)^2 - (8 - m)^2}{64} = \frac{(17 - 2m)}{64}$

(b) $P(M = 4) = \frac{9}{64}$

$E(M) = \Sigma\, m\, P(M = m) = \frac{1}{64} \Sigma\, m(17 - 2m)$

$= \frac{51}{16}$

Answer: (b) $P(M = 4) = \frac{9}{64}$, $E(M) = \frac{51}{16}$