Day 11 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Binomial expansion with partial fractions (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = \frac{7x - 1}{(1 - 3x)(1 + x)}$.

(a) Express f(x) in partial fractions.

(b) Hence expand f(x) in ascending powers of x up to x^3, stating the range of validity.

(a) $7x - 1 \equiv A(1 + x) + B(1 - 3x)$: $A = 1$, $B = -2$.

$f(x) = \frac{1}{1 - 3x} - \frac{2}{1 + x}$.

(b) $\frac{1}{1 + cx} = 1 - cx + c^2x^2 - c^3x^3 + \ldots$ (use $c = -3$, then $c = 1$).

Coefficients of x^j: $1(3)^j - 2(-1)^j$.

Valid for $|x| < \frac{1}{3}$ (both expansions).

Answer: (a) $A = 1$, $B = -2$ (b) $-1 + 5x + 7x^2 + 29x^3 + \ldots$; $|x| < \frac{1}{3}$

2 Without replacement: find the number of counters (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A bag has 6 red counters and some black counters, n in total.

Two are taken without replacement. P(both red) = $\frac{15}{28}$. Find n.

$P(\text{both red}) = \frac{6}{n} \times \frac{5}{(n - 1)} = \frac{30}{n(n - 1)}$

So $n(n - 1) = 30 \div \frac{15}{28} = 56$

$n^2 - n - 56 = 0$, so $(n - 8)(n + 7) = 0$

$n > 0$, so $n = 8$

Answer: $n = 8$

3 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact gradient of the curve $y = 2^{2x - 3}$
at the point where $x = 3$.

$y = e^{(2x - 3)\ln 2}$

$dy/dx = 2 \ln 2 \times 2^{2x - 3}$

At $x = 3$: exponent = 3, so $dy/dx = 2 \ln 2 \times 2^3$

Answer: $16\ln 2$

4 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{-6 - 2\sqrt{5}}{4 - \sqrt{5}}$ in the form $p + q\sqrt{5}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(4 + \sqrt{5})$.

Numerator: $(-6 - 2\sqrt{5})(4 + \sqrt{5}) = -34 - 14\sqrt{5}$.

Denominator: $4^2 - 1^2(5) = 11$.

So the fraction = $\frac{-34 - 14\sqrt{5}}{11}$.

Answer: $-\frac{34}{11} - \frac{14}{11}\sqrt{5}$

5 Hypothesis test for a normal mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The daily output of a machine is N(μ, 36) with known variance 36. The mean was claimed to be 55 units. A random sample of 49 has mean 55.5. Test at the 1% level whether the mean is greater than 55.

H_0: $\mu = 55$, H_1: $\mu > 55$. Under H_0, the sample mean ~ $N(55, \frac{36}{49})$

$z = \frac{55.5 - 55}{6 \div \sqrt{49}} = 0.583$

p-value = $P(Z > 0.583) = 0.280$

$0.280 > 1\%$, so do not reject H_0

There is insufficient evidence at the 1% level that the mean has changed as claimed

Answer: Do not reject H_0 (z = 0.583, p-value 0.280)

6 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^2 - 2x + 1$ is translated by $\begin{pmatrix}3\\3\end{pmatrix}$

and then a stretch, scale factor 2, parallel to the x-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix}3\\3\end{pmatrix}$: $y = f(x - 3) + 3 = x^2 - 8x + 19$.

Then a stretch, scale factor 2, parallel to the x-axis: replace x by $\frac{1}{2}x$: $y = f(\frac{1}{2}x - 3) + 3$.

Expand and simplify to give the quadratic.

Answer: $y = \frac{1}{4}x^2 - 4x + 19$

7 Identity then equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\cos^4\theta - \sin^4\theta \equiv \cos 2\theta$.

(b) Hence solve $\cos^4\theta - \sin^4\theta = -\frac{1}{2}$ for $0 \le \theta < 360°$ (1 d.p.).

$\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta) = \cos 2\theta \times 1$

$\cos 2\theta = -\frac{1}{2}$, so $2\theta = 120.0°, 240.0°, 480.0°, 600.0°$ $(0 \le 2\theta < 720°)$

Halve each value

Answer: θ = 60.0°, 120.0°, 240.0°, 300.0°

8 Relative motion of two projectiles (AQA, Edexcel, OCR, MEI, Eduqas)

Particles A and B are projected from a point O at the same instant, with initial velocities $(32\mathbf{i} + 36\mathbf{j})$ m s^{-1} and $(20\mathbf{i} + 31\mathbf{j})$ m s^{-1} respectively (i horizontal, j vertically upwards).

Both are still in flight after 5 s. Find (a) the distance AB at this time, (b) the angle AB makes with the horizontal, to 1 d.p.

Position of A: $32t\mathbf{i} + (36t - 4.9t^2)\mathbf{j}$; position of B: $20t\mathbf{i} + (31t - 4.9t^2)\mathbf{j}$

Subtracting, the $4.9t^2$ terms cancel: $AB = t(12\mathbf{i} + 5\mathbf{j})$

After 5 s: $AB = (60\mathbf{i} + 25\mathbf{j})$

Distance = $\sqrt{(60^2 + 25^2)} = 65$

The direction of AB is that of $(12\mathbf{i} + 5\mathbf{j})$, which does not depend on t: tan angle = $\frac{5}{12}$

Answer: (a) 65 m (b) 22.6°

9 Exponential times trigonometric (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$y = e^{-2x} \sin 3x$. Show that $dy/dx = e^{-2x}(-2 \sin 3x + 3 \cos 3x)$.
Find the x coordinate of the stationary point with the smallest positive x (3 s.f.), and its nature.

Product rule: $dy/dx = (-2)e^{-2x} \sin 3x + e^{-2x} \times 3 \cos 3x$
$dy/dx = 0$ requires $-2 \sin 3x + 3 \cos 3x = 0$

$\tan 3x = \frac{3}{2}$, smallest positive $3x = 0.9828$

$x = 0.328$
Second derivative test (or gradient change): a maximum
Answer: x = 0.328, maximum

10 Non-uniform rod: centre of mass and tilting (AQA, Edexcel, OCR, MEI)

A non-uniform rod AB of length 5 m and mass 4 kg rests on supports at C and D, where AC = 1 m and AD = 4 m, with $R_C = 2 R_D$.
Find (a) the distance of the centre of mass from A. A particle of mass m kg at B puts the rod on the point of tilting about D. (b) Find m.

$R_C = 2 R_D$. Moments about A: $R_C \times 1 + R_D \times 4 = 4g \times AG$

$R_D(2 \times 1 + 4) = 4g \times AG$ with $R_D(1 + 2) = 4g$, so $AG = \frac{2 \times 1 + 4}{3} = 2$

Tilting about D: $R_C = 0$. Moments about D: $m g (5 - 4) = 4 g (4 - 2)$

$m = \frac{4 \times 2}{1} = 8$

Answer: (a) 2 m (b) 8