Day 106 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Triangular pdf from a diagram (CIE)

The diagram shows the pdf of X.
Find (a) k, (b) E(X), (c) the median of X (3 s.f.).

(a) Area of triangle = ½ × 4 × k = 1, so k = $\frac{1}{2}$

(b) The mean of a triangular distribution is $\frac{0+1+4}{3} = \frac{5}{3}$

F(1) = $\frac{1}{4} = \frac{1}{4} < \frac{1}{2}$, so the median is in the right part

Right-hand area $\frac{(4-x)^2}{12} = \frac{1}{2}$, so $x = 4 - \sqrt{6} = 1.551$

Answer: (a) k = $\frac{1}{2}$ (b) E(X) = $\frac{5}{3}$ (c) 1.55

2 Related rates of change (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cone with its vertex downwards and semi-vertical angle α, where
$\tan\alpha = \frac{1}{2}$, is filled at 4π cm^{3}s^{-1}. Find the rate at which
the depth h is rising when h = 6 cm. ($V = \frac{1}{3}\pi r^2 h$)

$r = \frac{h}{2}$, so $V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h = \frac{\pi}{12}h^3$

$dV/dt = \frac{\pi}{4}h^2\, dh/dt$, so $4\pi = \frac{\pi}{4} \times 36 \times dh/dt$

$dh/dt = \frac{16}{36}$

Answer: $\frac{4}{9}$ cm s^{-1}

3 Geometric series: sum to infinity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric series has sum to infinity 16 and second term 3. Find the possible values of the common ratio r and the first term a in each case.

$a = \frac{3}{r}$. $S_\infty = \frac{a}{1-r} = \frac{3}{r(1-r)} = 16$.

$16r^2 - 16r + 3 = 0$.

$(4r-1)(4r-3) = 0$: $r = \frac{1}{4}$ or $r = \frac{3}{4}$.

Need $|r| < 1$: $\frac{1}{4}, \frac{3}{4}$ valid.

Answer: r = $\frac{1}{4}$, a = 12 or r = $\frac{3}{4}$, a = 4

4 Lift, passenger and cable (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 350 kg, carrying a passenger of mass 70 kg, is supported by a vertical cable. The lift is moving downwards and accelerating at 2.4 m s^{-2}.
Find (a) the force the floor exerts on the passenger, (b) the tension in the cable. Take g = 9.8 m s^{-2}.

Take the direction of the acceleration as positive: it is downwards
Passenger: $70 \times 9.8 - R = 70 \times 2.4$
$R = 70 \times (9.8 - 2.4) = 518$
Lift and passenger: $420 \times 9.8 - T = (350 + 70) \times 2.4$
$T = 420 \times 7.4 = 3108$
Answer: (a) 518 N (b) 3108 N

5 R-form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Express $12\cos\theta + 9\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$. Give α to 1 d.p.
(b) Hence solve $12\cos\theta + 9\sin\theta = 12$ for $0 \leq \theta < 360°$, giving answers to 1 d.p.

$R = \sqrt{12^2 + 9^2} = 15$ and $\tan\alpha = \frac{9}{12}$, so $\alpha = 36.9°$

So $12\cos\theta + 9\sin\theta \equiv 15\cos(\theta - 36.9°)$

$\cos(\theta - \alpha) = \frac{12}{15}$

$\theta - \alpha = \pm 36.9°$ (using $\alpha = 36.9°$)

Adding or subtracting multiples of 360° to give $0 \leq \theta < 360°$

Answer: (a) $R = 15$, $\alpha = 36.9°$ (b) $\theta = 0.0°, 73.7°$

6 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(8\mathbf{i} - 3\mathbf{j})$ km relative to a fixed origin and B is at $(6\mathbf{i} + 8\mathbf{j})$ km. A has velocity $(4\mathbf{i} + 2\mathbf{j})$ km h^{-1} and B has velocity $(7\mathbf{i} - 2\mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (-2\mathbf{i} + 11\mathbf{j}) + t(3\mathbf{i} - 4\mathbf{j})$

$|\mathbf{d}|^2 = (3t - 2)^2 + (-4t + 11)^2 = 25t^2 - 100t + 125$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-50}{25} = 2$

$\mathbf{d} = (-2\mathbf{i} + 11\mathbf{j}) + 2(3\mathbf{i} - 4\mathbf{j}) = (4\mathbf{i} + 3\mathbf{j})$, so least distance = 5

Answer: (a) $t = 2$ h (b) 5 km

7 Newton-Raphson iteration (AQA, Edexcel, OCR, MEI, Eduqas (Newton-Raphson))

$f(x) = x^3 + x^2 - x - 8$.
(a) Show that $f(x) = 0$ has a root α with $1 < \alpha < 2$.
(b) Use Newton-Raphson with $x_0 = 2$ to find x_1 and x_2
to 4 decimal places.

$f(1) = -7$ and $f(2) = 2$: a change of sign, f continuous

$f'(x) = 3x^2 + 2x - 1$, so $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = 2 - \frac{2.000}{15.00} = 1.8667$

$x_2 = 1.8667 - \frac{0.1221}{13.19} = 1.8574$

Answer: $x_1 = 1.8667$, $x_2 = 1.8574$

8 Parametric curve and line intersection (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has parametric equations
$x = 2t$, $y = 4t^2$. The line $y = -4x + 32$
meets C at A and B. Find the exact length AB.

Substitute: $4t^2 = -8t + 32$.

$4t^2 + 8t - 32 = 0 \Rightarrow t = 2$ or $t = -4$.

A(4, 16), B(−8, 64).

$AB^2 = 12^2 + (-48)^2 = 2448$.

Answer: $AB = 12\sqrt{17}$

9 Large data set: linear change of units (AQA, Edexcel, OCR, MEI, Eduqas)

Large Data Set, daily total rainfall (mm), Camborne, August 2015 (n = 92): mean 3.1, sd 6.8. (1 inch = 25.4 mm)
(a) Find the mean and sd in inches.
(b) Is a reading of 20.1 more than 2 sd above the mean?

New value = $\frac{5}{127}$ × old, so mean = $\frac{5}{127}$ × 3.1 = 0.1220

The constant shifts the mean but not the spread; only the multiplier affects σ

$\sigma = \frac{5}{127} \times 6.8 = 0.2677$

Outlier cut-off = mean + 2σ = 3.1 + 13.6 = 16.7 (original units)
20.1 is greater than 16.7, so yes
Answer: (a) mean 0.122, sd 0.268 (b) yes

10 Proof by contradiction (irrationality) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $\sqrt{5}$ is irrational, prove by contradiction that $9 + 2\sqrt{5}$ is irrational.

Assume $9 + 2\sqrt{5} = r$, where r is rational.

Then $\sqrt{5} = \frac{r - 9}{2}$.

$r - 9$ is rational and 2 is a non-zero integer,

so $\frac{r - 9}{2}$ is rational. This contradicts $\sqrt{5}$ irrational.

Hence $9 + 2\sqrt{5}$ is irrational.
Answer: Proved