Day 105 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Factor theorem and cubic (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = 3x^3 + ax^2 + bx + 27$, where a and b are constants.
$(x - 1)$ is a factor of f(x), and f(x) leaves remainder 45 when divided by $(x + 2)$.
Find a and b, and hence solve $f(x) = 0$.

$f(1) = 0$: $3 + a + b + 27 = 0$, so $a + b + 30 = 0$.
$f(-2) = 45$: $4a - 2b + 3 = 45$.
Solving: $a = -3$, $b = -27$.
$f(x) = 3x^3 - 3x^2 - 27x + 27 = 3(x + 3)(x - 1)(x - 3)$
Answer: $a = -3$, $b = -27$; $x = -3, 1, 3$

2 Independence from a Venn diagram (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The Venn diagram shows probabilities; A and B are independent.
Find the possible values of x.

$P(A) = 0.1 + x$, $P(B) = 0.45 + x$, $P(A \cap B) = x$
Independent: $x = (0.1 + x)(0.45 + x)$
$x^2 - 0.45x + 0.045 = 0$
$(x - 0.15)(x - 0.3) = 0$
Both values keep every probability between 0 and 1
Answer: $x = 0.15$ or $x = 0.3$

3 Inverse function and range (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The function f is defined by $f(x) = x^2 - 8x + 24$
for $x \geq k$. Given that k is the least value for which
f is one-one, find k, $f^{-1}(x)$, and the domain of f^{-1}.

Complete the square: $f(x) = (x - 4)^2 + 8$.
The vertex is at $x = 4$, so $k = 4$ (f is one-one on one side of it).
Let $y = f(x)$: $y - 8 = (x - 4)^2$, so $(x - 4) = \pm\sqrt{y - 8}$.
As $x \geq 4$, take the positive root: $x = 4 + \sqrt{y - 8}$.
Range of f = domain of f^{-1}: $f(4) = 8$, so $x \geq 8$.
Answer: $k = 4$; $f^{-1}(x) = 4 + \sqrt{x - 8}$; $x \geq 8$

4 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 3^x - 2x$ has one stationary point.
Find its coordinates to 3 s.f. and show that it is a minimum.

$dy/dx = 3^x \ln 3 - 2 = 0$
$3^x = \frac{2}{\ln 3}$, $x = \frac{1}{\ln 3} \ln(\frac{2}{\ln 3}) = 0.545$
$d^2y/dx^2 = 3^x(\ln 3)^2 > 0$ for all x: minimum
$y = \frac{2}{\ln 3} - 2 \times x = 0.730$
Answer: (0.545, 0.730)

5 Particle in equilibrium on two strings (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle P of weight 141 N hangs in equilibrium from two light strings AP and BP attached to a horizontal ceiling at A and B (see diagram), where $\tan\alpha = \frac{3}{4}$ and $\tan\beta = \frac{4}{3}$.

Find (a) the tension in each string, (b) the greatest weight P could have if neither tension may exceed 136.8 N.

Horizontally: $T_1 \cos\alpha = T_2 \cos\beta$, so $T_1 \times \frac{4}{5} = T_2 \times \frac{3}{5}$

Vertically: $T_1 \sin\alpha + T_2 \sin\beta = 141$, and $T_2 = \frac{4}{3} T_1$

$T_1(\frac{3}{5} + \frac{4}{3} \times \frac{4}{5}) = 141$, so $T_1 = 84.6$ and $T_2 = 112.8$

The tension in BP is the larger: it is 112.8 when W = 141

Tensions are proportional to W, so $W_{max} = 141 \times \frac{136.8}{112.8} = 171$

Answer: (a) AP: 84.6 N, BP: 112.8 N (b) 171 N

6 Parametric differentiation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3t^2 + 3t$, $y = 2t^3 - 4t$.
Find the equation of the tangent at the point where t = 2,
in the form y = mx + c.

$dx/dt = 6t + 3$, $dy/dt = 6t^2 - 4$

At t = 2: $dy/dx = \frac{20}{15} = \frac{4}{3}$; point (18, 8)

$y - 8 = \frac{4}{3}(x - 18)$

Answer: $y = \frac{4}{3}x - 16$

7 Screening test and Bayes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A rare allele is present in 2% of a population. A test is positive for 95% of those with it and for 5% of those without it.
Find P(positive) and P(has it | positive).

P(positive) = 0.02 × 0.95 + 0.98 × 0.05
= 0.019 + 0.049 = 0.068

$P(\text{has it} \mid \text{positive}) = \frac{0.019}{0.068}$

= 0.2794

Answer: (a) 0.068 (b) 0.279

8 Optimisation with justification (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

An open-topped box has a square base of side x m and volume 4 m^3.

Show that the surface area is $A = x^2 + \frac{16}{x}$, and find the
minimum value of A, justifying that it is a minimum.

$h = \frac{4}{x^2}$, $A = x^2 + 4xh = x^2 + \frac{16}{x}$

$dA/dx = 2x - \frac{16}{x^2} = 0$, so $x^3 = 8$, $x = 2$

$d^2A/dx^2 = 2 + \frac{32}{x^3} > 0$, so a minimum

$A = 4 + \frac{16}{2}$

Answer: 12 m^2

9 Tangent line to circle: finding k (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The line $3y = 4x + k$ is a tangent to the circle
$(x - 1)^2 + (y + 1)^2 = 9$.
Find the two possible values of k.

Substitute $y = \frac{4x + k}{3}$: $(x - 1)^2 + (\frac{4x + k}{3} + 1)^2 = 9$.

Multiply by 9 and collect: a quadratic in x with a = 25.

Tangent ⇒ discriminant = 0. This simplifies to $(k + 7)^2 = 225$.

$k + 7 = \pm 15$.

Answer: $k = -22$ or $k = 8$

10 Closest approach of two ships (AQA, Edexcel, OCR, MEI, Eduqas)

Two ships A and B move with constant velocities. At time t = 0 hours, A is at $(-2\mathbf{i} - 3\mathbf{j})$ km relative to a fixed origin and B is at $(5\mathbf{i} - 4\mathbf{j})$ km. A has velocity $(5\mathbf{i} + 2\mathbf{j})$ km h^{-1} and B has velocity $(2\mathbf{i} + 6\mathbf{j})$ km h^{-1}.
Find (a) the time at which the ships are closest together, (b) the least distance between them.

Position of B relative to A: $\mathbf{d} = (7\mathbf{i} - \mathbf{j}) + t(-3\mathbf{i} + 4\mathbf{j})$

$|\mathbf{d}|^2 = (-3t + 7)^2 + (4t - 1)^2 = 25t^2 - 50t + 50$

Least when $\frac{d}{dt}|\mathbf{d}|^2 = 0$: $t = -\frac{-25}{25} = 1$

$\mathbf{d} = (7\mathbf{i} - \mathbf{j}) + 1(-3\mathbf{i} + 4\mathbf{j}) = (4\mathbf{i} + 3\mathbf{j})$, so least distance = 5

Answer: (a) t = 1 h (b) 5 km