Day 104 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Projectile through a given point: two angles (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 35 m s^{-1}, at angle θ above the horizontal. It passes through the point P, 49 m horizontally and 26 m vertically from O.

Find the two possible values of $\tan\theta$, and the two possible values of θ to 1 d.p. Take g = 9.8 m s^{-2}.

Horizontal: $x = 35\cos\theta \times t$. Vertical: $y = 35\sin\theta \times t - 4.9t^2$

Eliminating t: $y = x\tan\theta - \frac{9.8x^2}{2u^2}(1 + \tan^2\theta)$

$\frac{9.8 \times 49^2}{2 \times 35^2} = 9.604$, so $26 = 49\tan\theta - 9.604(1 + \tan^2\theta)$

$9.604\tan^2\theta - 49\tan\theta + 35.604 = 0$

$\tan\theta = \frac{43}{49}$ or $\frac{207}{49}$, so $\theta = 41.3°$ or $76.7°$

Answer: $\tan\theta = \frac{43}{49}$ or $\frac{207}{49}$; $\theta = 41.3°$ or $76.7°$

2 Simultaneous equations (linear and quadratic) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve simultaneously:

$x - 2y = -2$

$x^2 + xy - y^2 = 19$

From the first equation $x = -2 + 2y$.

Substitute into the second equation and collect terms:

this gives $5y^2 - 10y - 15 = 0$.

$y = -1$ or $y = 3$, giving $x = -4$ or $x = 4$.

Answer: (x, y) = (−4, −1) and (4, 3)

3 Stone thrown from a cliff (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is thrown vertically upwards with speed 18 m s^{-1} from the top of a cliff. The top of the cliff is 32.5 m above the sea.

Find (a) the time taken for the stone to reach the sea, (b) the total distance travelled by the stone, to 3 s.f. Take g = 9.8 m s^{-2}.

Take upwards as positive: $-32.5 = 18t - 4.9t^2$, so $4.9t^2 - 18t - 32.5 = 0$

$t = \frac{18 + \sqrt{(18^2 + 19.6 \times 32.5)}}{9.8} = \frac{18 + 31}{9.8} = 5$

Height risen above the cliff top = $\frac{18^2}{(2 \times 9.8)} = 16.5$ m

Total distance = 2 × 16.5 + 32.5

Answer: (a) 5 s (b) 65.6 m

4 Fixed point iteration and rearrangement (AQA, Edexcel, OCR, MEI, Eduqas, CIE (iterative methods))

The equation $x^3 = 2x + 7$ has a root α.

(a) Show that the equation can be written as $x = \sqrt[3]{(2x + 7)}$.

(b) Use $x_{n+1} = \sqrt[3]{(2x_n + 7)}$ with $x_0 = 1$ to find x_1, x_2, x_3 to 4 d.p., and hence state α to 3 d.p.

Take the cube root of both sides to obtain $x = \sqrt[3]{(2x + 7)}$

$x_1 = 2.0801$, $x_2 = 2.2347$, $x_3 = 2.2552$

The iteration converges (successive values agree to 3 d.p.): $\alpha = 2.258$

Answer: $x_1 = 2.0801$, $x_2 = 2.2347$, $x_3 = 2.2552$; $\alpha = 2.258$

5 Cubic graph: find the equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the cubic y = f(x), which touches the x-axis once. Find f(x), giving your answer in expanded form.

Crosses at x = 2 and touches at x = −2, so $f(x) = k(x - 2)(x + 2)^2$.

At (0, −8): $-8 = k(-2)(2)^2 = -8k$, so k = 1.

$f(x) = (x - 2)(x + 2)^2$, then expand.

Answer: $f(x) = x^3 + 2x^2 - 4x - 8$

6 Large data set: linear change of units (AQA, Edexcel, OCR, MEI, Eduqas)

Large Data Set, daily total rainfall (mm), Camborne, June 2015 (n = 184): mean 3.7, sd 4.8. (1 inch = 25.4 mm)
(a) Find the mean and sd in inches.
(b) Is a reading of 9.9 more than 2 sd above the mean?

New value = $\frac{5}{127}$ × old, so mean = $\frac{5}{127}$ × 3.7 = 0.1457

The constant shifts the mean but not the spread; only the multiplier affects σ

σ = $\frac{5}{127}$ × 4.8 = 0.1890

Outlier cut-off = mean + 2σ = 3.7 + 9.6 = 13.3 (original units)
9.9 is less than 13.3, so no

Answer: (a) mean 0.146, sd 0.189 (b) no

7 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (2, 0, 5), (3, −2, 4), (4, 0, 3).
Find angle ABC to 1 d.p., and the exact area of triangle ABC.

BA = $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, BC = $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$; BA · BC = 2

|BA| = $\sqrt{6}$, |BC| = $\sqrt{6}$

cos ABC = $\frac{2}{\sqrt{6}\sqrt{6}}$ = 0.3333, so ABC = 70.5°

|BA × BC| = $\left|\begin{pmatrix} -4 \\ 0 \\ -4 \end{pmatrix}\right|$ = $\sqrt{32}$

Area = $\frac{1}{2}$|BA × BC|

Answer: ABC = 70.5°, area = $2\sqrt{2}$

8 Expected winnings and a fair game (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A game costs £3. Two fair dice are rolled: a total of at least 10 wins £14, a total of at most 3 wins £3, otherwise nothing.
(a) Find the expected profit.
(b) Find the standard deviation of the profit (3 s.f.).
(c) Find the stake that makes the game fair.

P(total ≥ 10) = $\frac{1}{6}$, P(total ≤ 3) = $\frac{1}{12}$, otherwise $\frac{3}{4}$

Profit: £11 with $\frac{1}{6}$, £0 with $\frac{1}{12}$, £−3 with $\frac{3}{4}$

(a) E = $-\frac{5}{12}$

(b) E(P^2) = $\frac{323}{12}$, Var = $\frac{323}{12} - (-\frac{5}{12})^2 = \frac{3851}{144}$, so σ = 5.171

(c) Fair when the stake equals E(winnings) = $\frac{1}{6} \times 14 + \frac{1}{12} \times 3 = \frac{31}{12}$

Answer: (a) $-£\frac{5}{12}$ (b) 5.17 (c) $£\frac{31}{12}$

9 Recurrence: limit and convergence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = \frac{1}{4}u_n + \frac{27}{2}$, $u_1 = 12$.

(a) Find the limit L of the sequence.

(b) Find the least n such that $|u_n - L| < 0.001$.

(a) At the limit $L = \frac{1}{4}L + \frac{27}{2}$, so $L(1 - \frac{1}{4}) = \frac{27}{2}$.

$L = 18$.

(b) $u_n - L = (u_1 - L)(\frac{1}{4})^{n-1} = (-6)(\frac{1}{4})^{n-1}$.

Need $6 \times (\frac{1}{4})^{n-1} < 0.001$, so $n - 1 > 6.275$.

The least n is 8.

Answer: (a) L = 18 (b) n = 8

10 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - 2xy + 5y^2 = 320$.
Find the coordinates of the points on C where the tangent is parallel to the y-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -2y}{-2x + 10y}$

Vertical tangent when $-2x + 10y = 0$, so $x = \frac{10y}{2}$

Substitute into the equation of C: $x^2 + (-2xy) + (5y^2) = 320$
Solve for the remaining coordinate

Answer: (−20, −4) and (20, 4)