Day 102 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Poisson: scaling the rate, then binomial (CIE)

Calls arrive at a switchboard at an average rate of 6 per hour, independently and at random.
(a) Find the probability of at least 3 in a 20-minute period.
(b) Find the probability that, of 7 separate 20-minute periods, exactly 3 have no arrivals.

20-minute period: $\lambda = 6 \times \frac{20}{60} = 2$, so $X \sim Po(2)$

(a) $P(X \geq 3) = 1 - P(X \leq 2) = 0.3233$

(b) $P(X = 0) = e^{-2} = 0.1353$

$Y \sim B(7, 0.1353)$: $P(Y = 3) = 0.0485$

Answer: (a) 0.323 (b) 0.0485

2 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Show that $\frac{5+\sqrt{5}}{5-\sqrt{5}} - \frac{5-\sqrt{5}}{5+\sqrt{5}}$

can be written in the form $k\sqrt{5}$, and find k.

Common denominator $(5 - \sqrt{5})(5 + \sqrt{5}) = 25 - 5 = 20$.

Numerator: $(5 + \sqrt{5})^2 - (5 - \sqrt{5})^2$

$= (25 + 5 + 25\sqrt{5}) - (25 + 5 - 25\sqrt{5}) = 20\sqrt{5}$.

So the expression $= \frac{20\sqrt{5}}{20}$.

Answer: k = 1

3 Particle on a smooth slope joined to hanging particle (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Particles A (9 kg) and B (9 kg) are joined by a light string over a smooth pulley (see diagram). A lies on a smooth slope, $\alpha = 30°$, and B is 3.5 m above the ground.
Released from rest, find the further distance A moves up the slope after B lands (A does not reach the pulley).

A: $T - 9 \times 9.8 \times \frac{1}{2} = 9a$. B: $9 \times 9.8 - T = 9a$

Adding: $9.8(9 - 9 \times \frac{1}{2}) = 18a$, so $a = 2.45$

$T = 9(9.8 - 2.45) = 66.15$

When B lands: $v^2 = 2 \times 2.45 \times 3.5 = 17.15$

Then A decelerates at $g \sin \alpha = 4.9$: further distance $= \frac{17.15}{2 \times 4.9} = 1.75$

Answer: 1.75 m

4 Logarithmic equations with ln (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2 \ln x = \ln(x + 10) + \ln 5$,
stating clearly why one root is rejected.

$\ln x^2 = \ln(5(x + 10))$, so $x^2 = 5x + 50$

$x^2 - 5x - 50 = 0$, $(x - 10)(x + 5) = 0$

$x = -5$ is rejected since $\ln x$ is undefined for $x < 0$

Answer: x = 10

5 Beam on two supports: where can a person stand (AQA, Edexcel, OCR, MEI)

A uniform plank AB of length 8 m and mass 30 kg rests on supports at C and D, where AC = 2 m and DB = 3 m. A girl of mass 40 kg stands on it at distance x m from A.
Find the range of x for which the plank stays in equilibrium on both supports.

Equilibrium on both supports needs $R_C \geq 0$ and $R_D \geq 0$. Limiting cases: $R_C = 0$ and $R_D = 0$ in turn

Tilting about D (girl beyond D): $30(1) = 40(x - 5)$, so $x = 5 + \frac{30 \times 1}{40} = 5.75$

Tilting about C (girl before C): $30(2) = 40(2 - x)$, so $x = 2 - \frac{30 \times 2}{40} = 0.5$

Answer: $0.5 \leq x \leq 5.75$

6 Large data set: linear change of units (AQA, Edexcel, OCR, MEI, Eduqas)

Large Data Set, daily mean air temperature (°C), Hurn, October 2015 (n = 31): mean 16.9, sd 2.3. (F = 1.8C + 32)
(a) Find the mean and sd in °F.
(b) Is a reading of 20.8 more than 2 sd above the mean?

New value = 1.8 × old + 32, so mean = 1.8 × 16.9 + 32 = 62.42
The constant shifts the mean but not the spread; only the multiplier affects σ
σ = 1.8 × 2.3 = 4.140
Outlier cut-off = mean + 2σ = 16.9 + 4.6 = 21.5 (original units)
20.8 is less than 21.5, so no
Answer: (a) mean 62.4, sd 4.14 (b) no

7 Binomial approximation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Expand $(8 + x)^{\frac{1}{3}}$ up to and including the term in x^2.
(b) Use x = 0.01 to estimate $\sqrt[3]{8.01}$ to 5 decimal places.

(a) $(8 + x)^{\frac{1}{3}} = 2(1 + \frac{1}{8}x)^{\frac{1}{3}}$.

$= 2 + \frac{1}{12}x - \frac{1}{288}x^2 + ...$

(b) 8 + x with x = 0.01 gives 8.01.

Substitute: $2 + \frac{1}{12}(0.01) + (-\frac{1}{288})(0.01)^2$.

= 2.00083 (5 d.p.).

Answer: (a) $2 + \frac{1}{12}x - \frac{1}{288}x^2$ (b) 2.00083

8 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is obtuse. $\cos A = \frac{20}{29}$ and $\cos B = -\frac{3}{5}$.

Find the exact values of tan(A + B) and tan(A − B).

$\cos A = \frac{20}{29}$, $\sin A = \frac{21}{29}$; $\sin B = \frac{4}{5}$, $\cos B = -\frac{3}{5}$

$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\frac{17}{145}}{-\frac{144}{145}} = -\frac{17}{144}$

$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{-\frac{143}{145}}{\frac{24}{145}} = -\frac{143}{24}$

Answer: $\tan(A + B) = -\frac{17}{144}$, $\tan(A - B) = -\frac{143}{24}$

9 Rational inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\frac{-2x - 6}{-2x - 4} \geq 2$.

Rearrange: $\frac{-2x - 6}{-2x - 4} - 2 \geq 0$.

Combine: $\frac{2x + 2}{-2x - 4} \geq 0$.

Critical values: x = −1 (zero of numerator), x = −2 (excluded, denominator zero).
Sign analysis of the three regions gives the solution.
Answer: $-2 < x \leq -1$

10 Partial fractions then integrate (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

(a) Show that $\frac{x^2 + 2x - 11}{(x + 3)(x + 5)} \equiv 1 + \frac{A}{x + 3} + \frac{B}{x + 5}$.

(b) Hence find the exact value of $\int_0^3 \frac{x^2 + 2x - 11}{(x + 3)(x + 5)}\,dx$.

Numerator = (x + 3)(x + 5) + A(x + 5) + B(x + 3)
x = −3 gives A = −4; x = −5 gives B = −2
Integral = [x + −4 ln(x + 3) − 2 ln(x + 5)]
Substitute limits 0 and 3
Answer: A = −4, B = −2; 3 − 10ln 2 + 2ln 5