Day 101 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Three-stage journey (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A train accelerates uniformly from rest at 0.7 m s^{-2} to a speed of V m s^{-1}, travels at this speed, then decelerates uniformly at 1.4 m s^{-2} to rest.
The journey takes 60 s and covers 630 m. Find V and the time spent at constant speed.

Time accelerating = $\frac{V}{0.7}$, time decelerating = $\frac{V}{1.4}$, so time at constant speed = $60 - \frac{V}{0.7} - \frac{V}{1.4}$

Area under the speed-time graph: $630 = 60V - \frac{V^2}{1.4} - \frac{V^2}{2.8}$

Multiply by 2ab: $2.1V^2 - 117.6V + 1234.8 = 0$
$V = 14$ or $V = 42$
$V = 42$ gives time at constant speed -30 s, which is negative, so $V = 14$
Time at constant speed = $60 - 20 - 10 = 30$ s
Answer: V = 14, 30 s

2 Arithmetic series from a sum formula (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

An arithmetic sequence has $u_5 = 19$ and $S_{12} = 318$.
(a) Find the first term and common difference.
(b) Find the least n for which $S_n > 636$.

u_5: $a + 4d = 19$. S_{12}: $12(2a + 11d)/2 = 318$, so $12a + 66d = 318$.
Solving: $a = -1$, $d = 5$.
$S_n = \frac{n}{2}(-2 + 5n - 5) = \frac{n(5n - 7)}{2}$.
Test: $S_{16} = 584 \leq 636$, $S_{17} = 663 > 636$.
Answer: (a) a = −1, d = 5 (b) n = 17

3 Velocity-time graph with a return (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The diagram shows the velocity-time graph of a particle P moving on a straight line (velocity in m s^{-1}, time in s). At time 9 s, P is back at its starting point.
Find (a) the value of W, (b) the greatest distance of P from its starting point.

Net area = 0 (back at start): $\frac{1}{2} \times 2 \times 4 + \frac{1}{2}(4 - W)(4) - W(3) = 0$
$4 + 8 - 2W - 3W = 0$, so $5W = 12$
$W = 2.4$

v = 0 where the line from (2, 4) to (6, −2.4) crosses the axis: $t = 2 + 4 \times \frac{4}{6.4} = 4.5$

Greatest distance = area of triangle = $\frac{1}{2} \times 4.5 \times 4$
Answer: (a) W = 2.4 (b) 9 m

4 Quadratic sequence from recurrence (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$u_{n+1} = u_n + 4n - 1$, $u_1 = -4$.
(a) Show that $u_n = an^2 + bn + c$ and find a, b, c.
(b) Find the least n for which $u_n > 323$.

$u_n = u_1 + \sum_{k=1}^{n-1}(4k - 1) = -4 + 4 \times \frac{(n - 1)n}{2} - 1(n - 1)$.

$= 2n^2 - 3n - 3$.
So $a = 2$, $b = -3$, $c = -3$.
$u_{13} = 296 \leq 323$ and $u_{14} = 347 > 323$.
Answer: (a) a = 2, b = −3, c = −3 (b) n = 14

5 Triple angle equations (AQA, Edexcel, OCR, MEI, Eduqas, CIE (triple angle))

Given that $\cos 3\theta \equiv 4\cos^3\theta - 3\cos\theta$, solve
$\cos 3\theta = 0$ for $0 < \theta < \pi$, in terms of π.

Substitute: $\cos\theta(4\cos^2\theta - 3 - 0) = 0$

$\cos\theta = 0$ or $\cos^2\theta = \frac{3}{4}$

$\cos\theta = 0$ gives $\frac{\pi}{2}$

$\cos^2\theta = \frac{3}{4}$ gives $\frac{\pi}{6}, \frac{5\pi}{6}$

Answer: $\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$

6 Compound interest and savings (GP) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A ball is dropped from a height of 4 m. After each bounce it rises to $\frac{4}{5}$ of its previous height.

(a) Find the total distance it travels before it stops.
(b) After how many bounces does it rise less than 0.1 m?

(a) Down H, then up and down each time: $H + 2Hr + 2Hr^2 + ...$

$= H + \frac{2Hr}{1 - r} = 4 + \frac{2 \times 4 \times \frac{4}{5}}{1 - \frac{4}{5}} = 36.$

(b) Rebound height after n bounces $= 4(\frac{4}{5})^n < 0.1$.

$n > \frac{\log(0.025)}{\log(0.8)} = 16.53$, so n = 17.

Answer: (a) 36 m (b) 17 bounces

7 Implicit differentiation: tangent or normal (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x e^y + y = 1$.
Find the equation of the tangent to C at the point (1, 0),
in the form $ax + by + c = 0$ with integers a, b, c.

Differentiate implicitly with respect to x

At (1, 0): $dy/dx = -\frac{1}{2}$

$y - 0 = (-\frac{1}{2})(x - 1)$, then clear fractions

Answer: $x + 2y - 1 = 0$

8 Tree diagram with an unknown probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The tree diagram shows two games played in turn.

Given the second is a H, P(first was a H) = $\frac{1}{2}$. Find x.

P(H then H) = 0.6x, P(second H) = 0.6x + 0.2(1 − x)

P(first H | second H) = $\frac{0.6x}{0.6x + 0.2(1 - x)} = \frac{1}{2}$

$0.6x = \frac{1}{2}(0.6x + 0.2 - 0.2x)$

Solve the linear equation for x: $x = \frac{1}{4}$

Answer: $x = \frac{1}{4}$

9 Implicit curves: horizontal and vertical tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve C has equation $x^2 - 3xy + 5y^2 = 220$.
Find the coordinates of the points on C where the tangent is parallel to the y-axis.

Differentiate: $2x + by + (bx + 2cy)dy/dx = 0$, so $dy/dx = -\frac{2x + -3y}{-3x + 10y}$

Vertical tangent when $-3x + 10y = 0$, so $x = \frac{10y}{3}$

Substitute into the equation of C: $x^2 + (-3xy) + (5y^2) = 220$
Solve for the remaining coordinate
Answer: $(-20, -6)$ and $(20, 6)$

10 Poisson hypothesis test (CIE)

Faults in a cable occur at random at a mean rate of 4 per hour. After a modification, 12 faults are found in 1 hour. Test at the 5% level whether the rate of faults has increased.

H_0: $\lambda = 4$ per hour, H_1: $\lambda > 4$ per hour
Over 1 hour: $X \sim Po(4)$
$P(X \geq 12) = 1 - P(X \leq 11) = 0.000915$
$0.000915 < 5\%$, so reject H_0
There is sufficient evidence that the rate has increased
Answer: Reject H_0 (p-value 0.000915)