Day 100 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Proof by contradiction (logarithms) (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Prove by contradiction that $\log_6 10$ is irrational.

Assume $\log_6 10 = \frac{m}{n}$, m, n positive integers.

Then $6^{\frac{m}{n}} = 10$, so $6^m = 10^n$.

Prime factors of 6^m and 10^n must be identical,

but 5 divides 10^n but 5 does not divide 6^m.

This is a contradiction, so $\log_6 10$ is irrational.

Answer: Proved

2 Logarithmic equation, reject a root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $\log_2(x + 1) + \log_2(x - 1) = 3$,

justifying clearly any root that you reject.

$(x + 1)(x - 1) = 2^3 = 8$

$x^2 - 9 = 0$

$(x - 3)(x + 3) = 0$, so $x = 3$ or $x = -3$

$x = -3$ makes $x + 1 = -2 < 0$ and log of a negative number is undefined

Answer: $x = 3$

3 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^2 - x - 1$ is translated by $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$

and then a stretch, scale factor 3, parallel to the x-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$: $y = f(x + 2) + 3 = x^2 + 3x + 4$.

Then a stretch, scale factor 3, parallel to the x-axis: replace x by $\frac{1}{3}x$: $y = f(\frac{1}{3}x + 2) + 3$.

Expand and simplify to give the quadratic.

Answer: $y = \frac{1}{9}x^2 + x + 4$

4 Differentiating a^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = 2^x - 3x$ has one stationary point.

Find its coordinates to 3 s.f. and show that it is a minimum.

$dy/dx = 2^x \ln 2 - 3 = 0$

$2^x = \frac{3}{\ln 2}$, $x = \frac{1}{\ln 2} \ln(\frac{3}{\ln 2}) = 2.11$

$d^2y/dx^2 = 2^x(\ln 2)^2 > 0$ for all x: minimum

$y = \frac{3}{\ln 2} - 3 \times x = -2.01$

Answer: (2.11, −2.01)

5 Surds: rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Express $\frac{1 - 3\sqrt{3}}{4 - \sqrt{3}}$ in the form $p + q\sqrt{3}$, where p and q are rational numbers.

Multiply top and bottom by the conjugate $(4 + \sqrt{3})$.

Numerator: $(1 - 3\sqrt{3})(4 + \sqrt{3}) = -5 - 11\sqrt{3}$.

Denominator: $4^2 - 1^2(3) = 13$.

So the fraction = $\frac{-5 - 11\sqrt{3}}{13}$.

Answer: $-\frac{5}{13} - \frac{11}{13}\sqrt{3}$

6 Addition formulae with exact values (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A is acute and B is obtuse. $\cos A = \frac{7}{25}$ and $\cos B = -\frac{21}{29}$.

Find the exact values of $\tan(A - B)$ and $\cos(A - B)$.

$\cos A = \frac{7}{25}$, $\sin A = \frac{24}{25}$; $\sin B = \frac{20}{29}$, $\cos B = -\frac{21}{29}$

$\tan(A - B) = \frac{\sin(A - B)}{\cos(A - B)} = \frac{-\frac{644}{725}}{\frac{333}{725}} = -\frac{644}{333}$

$\cos(A - B) = \cos A \cos B + \sin A \sin B = \frac{333}{725}$

Answer: $\tan(A - B) = -\frac{644}{333}$, $\cos(A - B) = \frac{333}{725}$

7 Screening test and Bayes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A disease is present in 5% of a population. A test is positive for 90% of those with it and for 10% of those without it.

Find P(positive) and P(has it | positive).

P(positive) = 0.05 × 0.9 + 0.95 × 0.1

= 0.045 + 0.095 = 0.14

P(has it | positive) = $\frac{0.045}{0.14}$

= 0.3214

Answer: (a) 0.14 (b) 0.321

8 Overtaking with a delayed start (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Car A passes a point O travelling at a constant speed of 12.1 m s^{-1}. 4 s later, car B leaves O from rest with constant acceleration 3 m s^{-2}, travelling in the same direction.

B overtakes A at the point X. Find (a) the time B takes to reach X, (b) the distance OX, (c) the speed of B at X.

Let B travel for t s. A: distance = 12.1(t + 4); B: distance = ½ × 3 × t^2

Equate: $3t^2 - 24.2t - 96.8 = 0$, so $(t - 11)(3t + 8.8) = 0$

t = 11 (the other root is negative)

OX = 3 × 11^2 / 2 = 181.5 m, and speed = 3 × 11 = 33

Answer: (a) 11 s (b) 181.5 m (c) 33 m s^{-1}

9 Rejected batches: binomial inside binomial (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A box holds 7 components, each faulty with probability 0.15, independently. A box is rejected if it has at least 3 faulty components.

(a) Find the probability that a box is rejected.

(b) 9 boxes are inspected. Find the probability that at least 3 are rejected.

Let X ~ B(7, 0.15). P(rejected) = $P(X \geq 3) = 1 - P(X \leq 2) = 0.07377$

Let Y = number of rejected boxes, Y ~ B(9, 0.07377)

$P(Y \geq 3) = 1 - P(Y \leq 2) = 0.0240$

Answer: (a) 0.0738 (b) 0.0240

10 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(-8\mathbf{i} + 36\mathbf{j})$ km relative to a port O, and moves with constant velocity $(-3\mathbf{i} + 3\mathbf{j})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 13 km h^{-1} and intercepts A.

Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(-8\mathbf{i} + 36\mathbf{j}) + t(-3\mathbf{i} + 3\mathbf{j})$. B at time t: t v_B (unknown velocity)

At interception B has travelled 13 t, so $|(-8\mathbf{i} + 36\mathbf{j}) + t(-3\mathbf{i} + 3\mathbf{j})| = 13\,t$

Squaring and rearranging: $-151t^2 + 264t + 1360 = 0$, which gives t = 4 or a negative value

t = 4 (the other root is negative since B is faster than A)

A is then at $(-8\mathbf{i} + 36\mathbf{j}) + 4(-3\mathbf{i} + 3\mathbf{j}) = (-20\mathbf{i} + 48\mathbf{j})$, so $v_B = \frac{1}{4}(-20\mathbf{i} + 48\mathbf{j}) = (-5\mathbf{i} + 12\mathbf{j})$

Answer: (a) 4 h (b) $(-5\mathbf{i} + 12\mathbf{j})$ km h^{-1}