Day 1 **A Level Maths: S Tier 10-a-day: Model Solutions**

1 Modulus inequalities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $|3x - 3| > x + 5$.

Boundary values: $3x - 3 = x + 5$ gives $x = 4$;

$3x - 3 = -(x + 5)$ gives $x = -\frac{1}{2}$.

Also the corner of $|3x - 3|$ is at $x = 1$.

Testing each region (check the graph of each side) gives the solution set.

Answer: $x < -\frac{1}{2}$ or $x > 4$

2 Poisson: scaling the rate, then binomial (CIE)

Emails arrive in an inbox at an average rate of 4 per hour, independently and at random.

(a) Find the probability of at least 3 in a 30-minute period.

(b) Find the probability that, of 10 separate 30-minute periods, exactly 3 have no arrivals.

30-minute period: $\lambda = 4 \times \frac{30}{60} = 2$, so $X \sim \text{Po}(2)$

(a) $P(X \geq 3) = 1 - P(X \leq 2) = 0.3233$

(b) $P(X = 0) = e^{-2} = 0.1353$

$Y \sim B(10, 0.1353)$: $P(Y = 3) = 0.107$

Answer: (a) 0.323 (b) 0.107

3 Derivative of the inverse function (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve is given by $x = e^y + y$.

Find dy/dx and d^2y/dx^2 when $y = 0$.

$dx/dy = e^y + 1 = 2$ when $y = 0$, so $dy/dx = \frac{1}{dx/dy} = \frac{1}{2}$

$d^2x/dy^2 = 1$ when $y = 0$

$d^2y/dx^2 = -\frac{d^2x}{dy^2/(dx/dy)^3} = -\frac{1}{2^3}$

Answer: $dy/dx = \frac{1}{2}$, $d^2y/dx^2 = -\frac{1}{8}$

4 Rough slope: up then back down (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle is projected at 15 m s^{-1} up a line of greatest slope of a rough plane inclined at α to the horizontal, where $\sin\alpha = \frac{3}{5}$ and $\mu = 0.25$.

Find (a) the distance it travels up the plane, to 3 s.f., (b) its speed on returning to its starting point. Take $g = 9.8$ m s^{-2}.

$\cos\alpha = \frac{4}{5}$. Up the plane: $R = mg\cos\alpha$, so $a_{up} = g(\sin\alpha + \mu\cos\alpha) = 9.8(\frac{3}{5} + 0.25 \times \frac{4}{5}) = 7.84$

$0 = 15^2 - 2 \times 7.84 \times d$, so $d = \frac{225}{15.68} = 14.35$

Down the plane: $a_{down} = g(\sin\alpha - \mu\cos\alpha) = 9.8(\frac{3}{5} - 0.25 \times \frac{4}{5}) = 3.92$

$v^2 = 2 \times 3.92 \times 14.349 = 112.5$, so $v = 10.6$

Answer: (a) 14.3 m (b) 10.6 m s^{-1}

5 Cubic with only one real root (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

$f(x) = x^3 - 9x + 28$.

Show that $(x + 4)$ is a factor of $f(x)$, and that the equation $f(x) = 0$ has exactly one real root.

$f(-4) = -64 + 36 + 28 = 0$, so $(x + 4)$ is a factor.

$f(x) = (x + 4)(x^2 - 4x + 7)$.

For $x^2 - 4x + 7 = 0$: $b^2 - 4ac = 16 - 28 = -12 < 0$.

No real roots from the quadratic, so the only real root is $x = -4$.

Answer: Proved; the only real root is $x = -4$

6 Stationary points: exact coordinates (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Find the exact coordinates of the stationary point of

$y = \frac{\ln x}{x^3}$ for $x > 0$,

and determine its nature.

$dy/dx = \frac{1 - 3\ln x}{x^4} = 0$ gives $\ln x = \frac{1}{3}$

$y = \frac{1}{3}e^{-1}$ when $x = \sqrt[3]{e}$

Second derivative at this point is negative, so it is a maximum

Answer: $(\sqrt[3]{e}, \frac{1}{3}e^{-1})$, maximum

7 Transformations: new equation (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The curve $y = x^2 - x + 6$ is translated by $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$

and then a stretch, scale factor 3, parallel to the x-axis.

Find the equation of the resulting curve in the form $y = ax^2 + bx + c$.

Translation by $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$: $y = f(x + 3) + 3 = x^2 + 5x + 15$.

Then a stretch, scale factor 3, parallel to the x-axis: replace x by $\frac{1}{3}x$: $y = f(\frac{1}{3}x + 3) + 3$.

Expand and simplify to give the quadratic.

Answer: $y = \frac{1}{9}x^2 + \frac{5}{3}x + 15$

8 Triangle in 3D: angle and area (CIE (9709); Further Maths on other boards)

The points A, B, C have coordinates (4, 1, 1), (1, 3, 5), (−1, −3, 0).

Find angle ABC to 1 d.p., and the exact area of triangle ABC.

$BA = \begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}$, $BC = \begin{pmatrix} -2 \\ -6 \\ -5 \end{pmatrix}$; $BA \cdot BC = 26$

$|BA| = \sqrt{29}$, $|BC| = \sqrt{65}$

$\cos ABC = \frac{26}{\sqrt{29}\sqrt{65}} = 0.5988$, so $ABC = 53.2°$

$|BA \times BC| = \left|\begin{pmatrix} -14 \\ 23 \\ -22 \end{pmatrix}\right| = \sqrt{1209}$

Area $= \frac{1}{2}|BA \times BC|$

Answer: ABC = 53.2°, area $= \frac{1}{2}\sqrt{1209}$

9 Interception at constant speed (AQA, Edexcel, OCR, MEI, Eduqas)

At time t = 0 hours, ship A is at the point with position vector $(52\mathbf{i} + 20\mathbf{j})$ km relative to a port O, and moves with constant velocity $(-\mathbf{i})$ km h^{-1}. Ship B leaves O at t = 0 with constant speed 13 km h^{-1} and intercepts A.

Find (a) the time at which B intercepts A, (b) the velocity of B as a vector.

Position of A at time t: $(52\mathbf{i} + 20\mathbf{j}) + t(-\mathbf{i})$. B at time t: t v_B (unknown velocity)

At interception B has travelled 13 t, so $|(52\mathbf{i} + 20\mathbf{j}) + t(-\mathbf{i})| = 13t$

Squaring and rearranging: $-168t^2 - 104t + 3104 = 0$, which gives t = 4 or a negative value

t = 4 (the other root is negative since B is faster than A)

A is then at $(52\mathbf{i} + 20\mathbf{j}) + 4(-\mathbf{i}) = (48\mathbf{i} + 20\mathbf{j})$, so $v_B = \frac{1}{4}(48\mathbf{i} + 20\mathbf{j}) = (12\mathbf{i} + 5\mathbf{j})$

Answer: (a) 4 h (b) $(12\mathbf{i} + 5\mathbf{j})$ km h^{-1}

10 One-tailed binomial test with data (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A manufacturer says that 10% of its screens have no flaws. A sample of 15 screens is checked, and 3 have no flaws.
Test, at the 5% significance level, the claim that the proportion is higher.
Give your conclusion in context.

H_0: p = 0.1, H_1: p > 0.1. Let X ~ B(15, 0.1)

$P(X \geq 3) = 1 - P(X \leq 2) = 0.184$

0.184 is greater than 5%, so do not reject H_0

There is insufficient evidence that the proportion is higher than the claimed 0.1

Answer: Do not reject H_0: p-value 0.184 > 5%