Day 312 **A Level Maths: B Tier 10-a-day: Model Solutions**

1 Projectile: greatest height and speed at the top (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A golf ball is projected from ground level with initial velocity components 19 m s^{-1} horizontally and 19.6 m s^{-1} vertically upwards.

Take g = 9.8 m s^{-2}.

Find (a) the greatest height, (b) the speed at the greatest height.

(a) $0 = 19.6^2 - 2 \times 9.8 \times H$, $H = 384.16 \div 19.6$

(b) At the top the vertical velocity is zero, so the speed is the horizontal component.

Answer: (a) 19.6 m (b) 19 m s^{-1}

2 Disproof by counter-example (AQA, Edexcel, OCR, MEI, Eduqas)

Statement: "4n + 13 is prime for every positive integer n."

Show that this statement is false by finding a counter-example.

Try n = 1, 2, 3, ... until the value is not prime.

n = 2 gives $4 \times 2 + 13 = 21$

$21 = 3 \times 7$, which is not prime.

Answer: n = 2 (value 21 is not prime)

3 Expectation and variance of aX + b (AQA, OCR, MEI, Eduqas, CIE)

The random variable X has E(X) = 10 and Var(X) = 5.

Find (a) E(4X + 5),

(b) Var(4X + 5), (c) $E(X^2)$.

(a) $E(aX + b) = aE(X) + b = 4 \times 10 + 5 = 45$

(b) $Var(aX + b) = a^2Var(X) = 16 \times 5 = 80$

(c) $Var(X) = E(X^2) - [E(X)]^2$, so $E(X^2) = 5 + 100 = 105$

Answer: (a) 45 (b) 80 (c) 105

4 Integrating velocity (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A particle moves in a straight line. At time t seconds its velocity is $v = 3t^2 + t$ m s^{-1}.

Find the distance moved between t = 1 and t = 5. (The particle does not change direction.)

$s = \int_1^5 (3t^2 + t)\,dt$

$= [\,t^3 + \frac{1}{2}t^2\,]$ from t = 1 to t = 5

$= 137.5 - 1.5$

Answer: 136 m

5 Exponential growth and decay models (All boards)

The mass of a radioactive sample, M grams, after t days is $M = 40\,e^{-0.1t}$.

Find (a) the half-life, (b) the rate of decrease in mass when t = 5, each to 3 s.f.

(a) $40\,e^{-0.1t} = 20$: $e^{-0.1t} = \frac{1}{2}$, so $t = \frac{\ln 2}{0.1} = 6.931$ days

(b) $\frac{dM}{dt} = -0.1 \times 40\,e^{-0.1t} = -4\,e^{-0.1t}$

At t = 5: $\frac{dM}{dt} = -4\,e^{-0.5} = -2.426$

so mass is decreasing at 2.43 g per day

Answer: (a) 6.93 days (b) 2.43 g/day

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6 Geometric series and logarithms (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A geometric sequence has first term 100 and common ratio 0.9.
Find the smallest n for which the nth term is less than 0.01.

$100 \times 0.9^{n-1} < 0.01$
$(n - 1) \lg 0.9 < \lg(0.01 \div 100)$
lg 0.9 is negative, so dividing reverses the inequality:
$n - 1 > \lg(0.01 \div 100) \div \lg 0.9 = 87.417$, so $n > 88.417$
The smallest integer n is 89
Answer: n = 89

7 Parametric differentiation: tangents (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A curve has parametric equations $x = 3 \sec t$, $y = 2 \tan t$.
Find $\frac{dy}{dx}$ in terms of t, and its exact value when $t = \frac{\pi}{4}$.

$\frac{dx}{dt} = 3 \tan t \sec t$, $\frac{dy}{dt} = 2 \tan^2 t + 2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 \operatorname{cosec} t}{3}$$

At $t = \frac{\pi}{4}$: $\frac{dy}{dx} = \frac{2\sqrt{2}}{3}$

Answer: $\frac{dy}{dx} = \frac{2 \operatorname{cosec} t}{3}$; value $\frac{2\sqrt{2}}{3}$

8 Proof by contradiction (AQA, Edexcel, OCR, MEI, Eduqas)

Prove by contradiction that there are no integers x and y such that $x^2 - y^2 = 6$.

Assume $x^2 - y^2 = 6$, so $(x - y)(x + y) = 6$.
$x - y$ and $x + y$ have the same parity (their sum 2x is even).
If both are odd, the product is odd; if both are even, the product is a multiple of 4.
But 6 is even and is not a multiple of 4 ($6 = 4 \times 1 + 2$), a contradiction.
So no such integers exist.
Answer: no integer solutions (product is odd or a multiple of 4)

9 Normal approximation to the binomial (AQA, OCR, CIE)

X ~ B(50, 0.6). Use a normal approximation to find $P(X \leq 31)$, to 3 significant figures.

$Y \sim N(np, np(1 - p)) = N(30.00, 12.00)$, so $\sigma = \sqrt{12.00} = 3.464$
Use a continuity correction.
$P(X \leq 31) \approx P(Y < 31.5) = P(Z < 0.4330) = 0.667$
Answer: 0.667

10 R cos(θ − α) form: solving (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Given that $5 \cos \theta + 12 \sin \theta \equiv 13 \cos(\theta - 67.4°)$,
solve $5 \cos \theta + 12 \sin \theta = -5$ for $0 \leq \theta \leq 360°$. (1 d.p.)

$13 \cos(\theta - 67.4°) = -5$, so $\cos(\theta - 67.4°) = \frac{-5}{13}$
$\theta - 67.4° = \pm 112.6°$
$\theta = 67.4° + 112.6° = 180.0°$
$\theta = 67.4° - 112.6° = -45.2°$, so $\theta = 314.8°$
Answer: θ = 180.0°, 314.8°