Day 83 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Arithmetic sequences: finding a and d (All boards)

The 2nd term of an arithmetic sequence is 1 and the 10th term is −31.
Find the first term and common difference, and the sum of the first 12 terms.

$u_{10} - u_2 = 8\,d = -31 - 1$ so $d = -4$

$a + 1\,d = 1$ so $a = 5$

$S_{12} = \frac{12}{2}(2 \times 5 + 11 \times (-4))$

Answer: a = 5, d = −4, S_{12} = −204

2 Continuous random variable: finding k and E(X) (CIE)

The random variable X has probability density function $f(x) = kx(3 - x)$ for $0 \le x \le 3$, and 0 otherwise.
(a) Find k. (b) Find E(X). (c) Find P(X > 2).

(a) $\int_0^3 f(x)\,dx = 1$ gives $\frac{9}{2}k = 1$, so $k = \frac{2}{9}$

(b) $E(X) = \int x\,f(x)\,dx = \frac{3}{2}$

(c) $P(X > 2) = \int_2^3 f(x)\,dx = \frac{7}{27}$

Answer: (a) k = $\frac{2}{9}$ (b) $\frac{3}{2}$ (c) $\frac{7}{27}$

3 Newton-Raphson method (AQA, Edexcel, OCR, MEI, Eduqas)

$f(x) = x^2 - 2$. Show that the Newton-Raphson formula gives $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n})$,

and use it with $x_0 = 2$ to find x_2 as an approximation to $\sqrt{2}$, to 4 d.p.

$f'(x) = 2x$, so $x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{1}{2}(x_n + \frac{2}{x_n})$

$x_1 = \frac{1}{2}(2 + \frac{2}{2}) = \frac{3}{2}$

$x_2 = 1.4167$

Answer: x_2 = 1.4167

4 Modelling with a normal distribution (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

At one location, the daily maximum gust is modelled by $N(21, 6^2)$.
(a) Find the probability that on a randomly chosen day the value exceeds 30 knots.
(b) Estimate the number of such days in a period of 31 days.

(a) $z = \frac{30 - 21}{6} = 1.50$

$P(Z > 1.50) = 0.0668$

(b) $31 \times 0.0668 = 2.07 \approx 2$

Answer: (a) 0.0668 (b) 2 days

5 Final seconds of a fall (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A stone is dropped from rest from the top of a cliff. In the last 3 seconds before it hits the sea it falls 147 m. Find the height of the cliff. Take g = 9.8.

Let the total time of fall be t s. Then $s = 4.9t^2$ for the whole fall.
In the first $(t - 3)$ s the stone falls $4.9(t - 3)^2$.
$4.9t^2 - 4.9(t - 3)^2 = 147$
$4.9(6t - 9) = 147$, so $t = 6$
Height $= 4.9 \times 6^2 = 207.025$
Answer: 207.025 m

6 Parametric differentiation: tangent and normal (All boards)

A curve has parametric equations $x = t^2 - 1$, $y = t^3 - 3t$.

Find $\frac{dy}{dx}$ in terms of t, and the equation of the normal at the point where $t = \frac{1}{2}$, in the form ax + by + c = 0 (integers).

$\frac{dy}{dx} = \frac{dy}{dt \div dx/dt} = \frac{3t^2 - 3}{2t}$

$t = \frac{1}{2}$: point $(-\frac{3}{4}, -\frac{11}{8})$, gradient $-\frac{9}{4}$

Normal gradient = $-1 \div (-\frac{9}{4}) = \frac{4}{9}$, so normal: $32x - 72y - 75 = 0$

Answer: $\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$**; $32x - 72y - 75 = 0$**

7 Triangle on a coordinate grid (All boards)

The diagram shows triangle ABC.

(a) Find the equation of the line through C perpendicular to AB.

(b) Hence find the coordinates of D, the foot of the perpendicular from C to AB, and the area of triangle ABC.

Gradient AB = 0, so gradient of the perpendicular = undefined (vertical)

Through C: x = 3

Line AB: $y - 2 = 0(x - -2)$. Solve to get D = (3, 2)

Area = ½ × AB × CD = 27

Answer: (a) x = 3 (b) D(3, 2), area 27

8 Intersecting or skew lines (CIE (9709); Further Maths on other boards)

l_1: $\mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$

l_2: $\mathbf{r} = \begin{pmatrix} -7 \\ 1 \\ 7 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

Determine whether l_1 and l_2 intersect. If they do, find the point of intersection.

Equate components: $-3 + 1\lambda = -7 - 2\mu$; $3 + 3\lambda = 1 - 1\mu$; $5 + 0\lambda = 7 + 1\mu$

Solving: $\lambda = 0$, $\mu = -2$, and all three equations are satisfied

Point of intersection = (−3, 3, 5)

Answer: Intersect at (−3, 3, 5)

9 Exponential equations: quadratic in b^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2 \times 3^{2x} - 24 \times 3^x + 54 = 0$.

Let $y = 3^x$. Then $2y^2 - 24y + 54 = 0$

$2(y - 9)(y - 3) = 0$

y = 9 or y = 3

$3^x = 9$ gives x = 2; $3^x = 3$ gives x = 1

Answer: x = 1 or x = 2

10 Range of horizontal force on a rough slope (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 2 kg particle is in equilibrium on a rough plane inclined at α, $\tan\alpha = \frac{12}{5}$, $\mu = \frac{1}{3}$. A horizontal force of P N, in the vertical plane through a line of greatest slope, pushes it towards the plane. Find the least and greatest values of P. Take g = 9.8. Give answers to 3 s.f. if not exact.

$\sin\alpha = \frac{12}{13}$, $\cos\alpha = \frac{5}{13}$, weight W = 19.6 N. Normal reaction $R = W\cos\alpha + P\sin\alpha$

Least P: about to slide down, friction $F = \mu R$ up the slope

$P\cos\alpha + \mu(W\cos\alpha + P\sin\alpha) = W\sin\alpha \rightarrow P = W(\sin\alpha - \mu\cos\alpha) \div (\cos\alpha + \mu\sin\alpha) = 22.5$

Greatest P: about to slide up, friction down the slope

$P\cos\alpha = W\sin\alpha + \mu(W\cos\alpha + P\sin\alpha) \rightarrow P = W(\sin\alpha + \mu\cos\alpha) \div (\cos\alpha - \mu\sin\alpha) = 268$

Answer: least 22.5 N, greatest 268 N