Day 70 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Connected rates of change: spheres, circles and cubes (AQA, Edexcel, OCR, MEI, Eduqas)

The volume of a spherical balloon increases at a constant rate of 30 cm^3s^{-1}.
Find the rates of increase of its radius and of its surface area when r = 5 cm.

$V = \frac{4}{3}\pi r^3$, so $\frac{dV}{dr} = 4\pi r^2$. $\frac{dr}{dt} = \frac{dV}{dt \div dV/dr} = \frac{30}{4\pi r^2}$

r = 5: $\frac{dr}{dt} = \frac{3}{10\pi}$ cm s^{-1}

$A = 4\pi r^2$, so $\frac{dA}{dr} = 8\pi r$ and $\frac{dA}{dt} = 8\pi \times 5 \times \frac{3}{10\pi} = 12$ cm^2s^{-1}

Answer: $\mathbf{\frac{dr}{dt} = \frac{3}{10\pi}}$ **cm/s,** $\mathbf{\frac{dA}{dt} = 12}$ $\mathbf{cm^2/s}$

2 Binomial probabilities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The probability that a seed germinates is 0.25. In 24 independent trials, find the probability that this happens (a) exactly 5 times (b) fewer than 5 times (c) at least 5 times.

Let X ~ B(24, 0.25)
(a) P(X = 5) = 0.176
(b) P(X ≤ 4) = 0.247
(c) 1 − P(X ≤ 4) = 0.753
Answer: (a) 0.176 (b) 0.247 (c) 0.753

3 Double angle equations (All boards)

Solve $\cos 2\theta + 5\cos\theta = -4$
for $-180° \le \theta \le 180°$.

$\cos 2\theta \equiv 2\cos^2\theta - 1$: $2\cos^2\theta + 5\cos\theta + 3 = 0$

Solve the quadratic: $\cos\theta = -1$ or $\cos\theta = -\frac{3}{2}$

$\cos\theta = -1$: $\theta = -180°, 180°$

$\cos\theta = -\frac{3}{2}$ is impossible

Answer: $\boldsymbol{\theta = -180°, 180°}$

4 Rationalising the denominator (AQA, Edexcel, OCR, MEI, Eduqas)

Express $\frac{2+\sqrt{5}}{5-2\sqrt{5}}$ in the form $p + q\sqrt{5}$, where p and q are rational numbers.

Multiply top and bottom by $5 + 2\sqrt{5}$.
Denominator: $(5 - 2\sqrt{5})(5 + 2\sqrt{5}) = 5$
Numerator: $(2 + \sqrt{5})(5 + 2\sqrt{5}) = 20 + 9\sqrt{5}$
So the expression is $\frac{20 + 9\sqrt{5}}{5}$.

Answer: $\mathbf{4 + \frac{9}{5}\sqrt{5}}$

5 Constant acceleration in vector form (AQA, Edexcel, OCR, MEI, Eduqas)

A particle P has constant acceleration $(-3\mathbf{i} + 2\mathbf{j})$ m s^{-2}. When t = 0, P is at $(8\mathbf{i} - 5\mathbf{j})$ m moving with velocity $(9\mathbf{i} + 4\mathbf{j})$ m s^{-1}. Find (a) when P moves parallel to $\mathbf{j}$, (b) its position vector then.

$v = u + at = (9\mathbf{i} + 4\mathbf{j}) + t(-3\mathbf{i} + 2\mathbf{j})$
(a) Parallel to $\mathbf{j}$: the $\mathbf{i}$ component is zero: $9 + -3t = 0$, so t = 3
(b) $r = r_0 + ut + \frac{1}{2}at^2$
$= (8\mathbf{i} - 5\mathbf{j}) + 3(9\mathbf{i} + 4\mathbf{j}) + 4.5(-3\mathbf{i} + 2\mathbf{j})$
$= 21.5\mathbf{i} + 16\mathbf{j}$
Answer: (a) t = 3 (b) r = (21.5i + 16j) m

6 Line meets curve: length of chord (All boards)

The line $y = 2x + 1$ meets the curve $y = x^2 + 3x - 5$ at A and B.
(a) Find the coordinates of A and B.
(b) Find the exact length AB.

$x^2 + 3x - 5 = 2x + 1$ gives $x^2 + x - 6 = 0$
$(x + 3)(x - 2) = 0$: $x = -3$ or 2
A$(-3, -5)$, B$(2, 5)$
$AB^2 = 5^2 + 10^2 = 125$
Answer: A$(-3, -5)$, B$(2, 5)$; AB = $5\sqrt{5}$

7 R-form and solving a sin x + b cos x = c (All boards)

(a) Express $12 \cos x + 9 \sin x$ in the form $R\cos(x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give α to 1 d.p.
(b) Solve $12 \cos x + 9 \sin x = -4$ for $0° \leq x < 360°$, giving answers to 1 d.p.

$R^2 = 12^2 + 9^2 = 225$, so $R = 15$; $\tan \alpha = \frac{9}{12}$ gives $\alpha = 36.9°$

$15 \cos(x - 36.9°) = -4$, so $\cos(x - 36.9°) = \frac{-4}{15}$

Principal value 105.5°, then use the symmetry of the cosine graph
$x = 142.3°, 291.4°$
Answer: (a) R = 15, α = 36.9° (b) x = 142.3°, 291.4°

8 Continuous random variable: variance (CIE)

X has pdf $f(x) = \frac{2}{25}x$ for $0 \leq x \leq 5$, and 0 otherwise.

Find E(X), Var(X) and the standard deviation of X (3 s.f.).

$E(X) = \int_0^5 x\, f(x)\, dx = \frac{10}{3}$

$E(X^2) = \int_0^5 x^2\, f(x)\, dx = \frac{25}{2}$

$\text{Var}(X) = \frac{25}{2} - (\frac{10}{3})^2 = \frac{25}{18}$

Standard deviation $= \sqrt{\frac{25}{18}} = 1.18$

Answer: E(X) = $\frac{10}{3}$, Var(X) = $\frac{25}{18}$, sd 1.18

9 Particles on two smooth inclined planes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The system is released from rest. Find (a) the magnitude of the acceleration and which particle moves down, (b) the tension. Take $g = 9.8$. Give answers to 3 s.f. if not exact.

A: weight component along plane = $3g \sin 60°$; B: $9g \sin 45°$
B moves down, A up ($9 \sin 45° > 3 \sin 60°$): F = ma for the system
$9g \sin 45° - 3g \sin 60° = (3 + 9)a$
(a) $a = 3.08$
(b) For A: $T - 3g \sin 60° = 3a$, so $T = 34.7$
Answer: (a) 3.08 m s^{-2}, B moves down (b) 34.7 N

10 Circle: centre, radius and position of a point (AQA, Edexcel, OCR, MEI, Eduqas)

A circle has equation $x^2 + y^2 - 4x + 6y - 16 = 0$.
(a) Find the centre and radius of the circle.
(b) Determine whether P$(2, -4)$ lies inside or outside the circle.

Complete the square: $(x - 2)^2 + (y + 3)^2 = 29$
Centre $(2, -3)$, radius $\sqrt{29}$
Distance^2 from centre to P $= 0^2 + (-1)^2 = 1$
Compare with $r^2 = 29$
Answer: (a) centre $(2, -3)$, radius $\sqrt{29}$ (b) P is inside the circle