Day 7 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Binomial expansion: a coefficient (All boards)

Find the coefficient of x^3 in the expansion of $(-2 - 2x)^8$.

Term in x^3: ${}^8C_3 \times (-2)^5 \times (-2x)^3$

$= 56 \times -32 \times -8$

Answer: 14336

2 Connected rates of change: spheres, circles and cubes (AQA, Edexcel, OCR, MEI, Eduqas)

A circular ripple has area A cm^2 increasing at a constant 6 cm^2s^{-1}.

Find the rate at which the circumference is increasing when the radius is 10 cm.

$A = \pi r^2$, so $\frac{dA}{dr} = 2\pi r$ and $\frac{dr}{dt} = \frac{6}{2\pi r}$

$C = 2\pi r$, so $\frac{dC}{dt} = 2\pi \times \frac{dr}{dt} = 2\pi \times \frac{6}{2\pi \times 10} = \frac{3}{5}$

Answer: $\frac{3}{5}$ cm/s

3 Systematic sampling (AQA, Edexcel, OCR, MEI, Eduqas)

A systematic sample is taken from a numbered list of 520 items.

The 4th item chosen is number 76 and the 7th item chosen is number 136.

Find the sampling interval, the number of the first item chosen and the size of the sample.

Interval = $(136 - 76) \div (7 - 4) = 20$

First item = $76 - 3 \times 20 = 16$

Sample size = $520 \div 20 = 26$

Answer: Interval 20, first 16, size 26

4 Inverse of a quadratic function (All boards)

$f(x) = 3x^2 - 18x + 30$, $x \geq 3$.

Find $f^{-1}(x)$ and state its domain.

$f(x) = 3(x - 3)^2 + 3$

Let $y = f(x)$: $(x - 3)^2 = \frac{y - 3}{3}$

$x \geq 3$, so take the positive root: $x = 3 + \sqrt{\frac{y - 3}{3}}$

Answer: $f^{-1}(x) = 3 + \sqrt{\frac{x - 3}{3}}$, $x \geq 3$

5 Lift and cable tension (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 250 kg carries a person of mass 75 kg. The cable tension is 3022.5 N and the lift is accelerating. Find (a) the magnitude and direction of the acceleration, (b) the force of the floor on the person. Take g = 9.8.

Whole system (mass 325 kg), taking upwards as positive: $T - 325g = 325a$

$3022.5 - 3185 = 325a$, so $a = -0.5$

(a) a is negative, so the acceleration is 0.5 m s^{-2} downwards

(b) Person: $R - 75g = 75a$, so $R = 75(9.8 - 0.5)$

Answer: (a) 0.5 m s^{-2} downwards (b) 697.5 N

6 Quotient rule: stationary points (All boards)

Find the coordinates of the stationary points of

$y = \frac{x}{x^2 + 9}$.

$\frac{dy}{dx} = \frac{(x^2 + 9) \times 1 - x \times 2x}{(x^2 + 9)^2} = \frac{9 - x^2}{(x^2 + 9)^2}$

$\frac{dy}{dx} = 0$ when $x^2 = 9$, so $x = \pm 3$

$x = 3$: $y = \frac{3}{18} = \frac{1}{6}$; $x = -3$: $y = -\frac{1}{6}$

Answer: $(-3, -\frac{1}{6})$, $(3, \frac{1}{6})$

7 Range of a composite function (All boards)

$f(x) = x^2 + 6x + 14$, $x \in \mathbb{R}$, and $g(x) = \frac{3}{x - 2}$, $x > 2$.

(a) Find the range of f.
(b) Show that gf is defined and find its range.

$f(x) = (x + 3)^2 + 5 \geq 5$
So $f(x) \geq 5 > 2$, which lies in the domain of g, so gf is defined.

$gf(x) = \frac{3}{f(x) - 2}$, with $f(x) \geq 5$, so $f(x) - 2 \geq 3$

Largest value is $\frac{3}{3} = 1$ at the minimum of f; $gf(x) > 0$.

Answer: (a) $f(x) \geq 5$ (b) $0 < gf(x) \leq 1$

8 Projectile from a cliff top (AQA, Edexcel, OCR, MEI, Eduqas)

A stone is projected from a cliff top 8.4 m above the sea at 35 m s^{-1}, at angle α above the horizontal, $\tan \alpha = \frac{4}{3}$. Find (a) the time to reach the sea, (b) the horizontal distance travelled, (c) the speed on impact. Take g = 9.8. Give answers to 3 s.f. if not exact.

Horizontal component 21, vertical component 28 (upwards)
(a) $-8.4 = 28\,t - 4.9t^2$, so $4.9t^2 - 28\,t - 8.4 = 0$
$t = [28 + \sqrt{948.64}] \div 9.8 = 6$ (taking the positive root)
(b) Distance = 21×6
(c) Vertical speed = $28 - 9.8 \times 6 = -30.8$; speed = $\sqrt{(21^2 + 30.8^2)}$

Answer: (a) 6 s (b) 126 m (c) 37.3 m s^{-1}

9 Foot of perpendicular and distance to a line (CIE (9709); Further Maths on other boards)

The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$.

Find the coordinates of the foot of the perpendicular from P(−2, −1, 4) to l, and the exact distance from P to l.

Foot F = A + t d: $\mathbf{PF} = \begin{pmatrix} 3 \\ 3t + 6 \\ -t - 2 \end{pmatrix}$

$\mathbf{PF} \cdot \mathbf{d} = 0$ gives $10t + 20 = 0$, so $t = -2$

F = (1, −1, 4); $\mathbf{PF} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$, so $|\mathbf{PF}| = \sqrt{9} = 3$

Answer: F(1, −1, 4), distance 3

10 Continuous random variable: cumulative distribution and median (CIE)

The random variable X has pdf $f(x) = \frac{3}{125}x^2$ for $0 \leq x \leq 5$, and 0 otherwise.

(a) Find the cumulative distribution function F(x) for $0 \leq x \leq 5$.

(b) Find $P(X > \frac{5}{2})$. (c) Find the median of X to 3 s.f.

(a) $F(x) = \int_0^x f(t)\,dt = \frac{x^3}{125}$

(b) $P(X > \frac{5}{2}) = 1 - F(\frac{5}{2}) = \frac{7}{8}$

(c) $F(m) = \frac{1}{2}$: $m^3 = \frac{5^3}{2}$, so $m = 5 \times (0.5)^{\frac{1}{3}}$

$m = 3.97$

Answer: (b) $\frac{7}{8}$ (c) 3.97