Day 69 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Optimisation problems (All boards)

A closed cylinder has radius r cm and volume 128π cm^3.

Show that its surface area is $A = 2\pi r^2 + \frac{256\pi}{r}$, and find the minimum value of A in terms of π.

$h = \frac{128}{r^2}$, so $A = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{256\pi}{r}$

$\frac{dA}{dr} = 4\pi r - \frac{256\pi}{r^2} = 0$, so $r^3 = 64$ and $r = 4$

$\frac{d^2A}{dr^2} = 4\pi + \frac{512\pi}{r^3} > 0$, so a minimum

$A = 2\pi \times 16 + \frac{256\pi}{4} = 96\pi$

Answer: A_min = 96π cm^2 (r = 4)

2 Forces in equilibrium with vectors (AQA, Edexcel, OCR, MEI, Eduqas)

Forces $F_1 = (-4\mathbf{i} - 4\mathbf{j})$ N, $F_2 = (5\mathbf{i} + 12\mathbf{j})$ N and $F_3 = (a\mathbf{i} + b\mathbf{j})$ N keep a 2 kg particle in equilibrium. Find (a) a and b. F_2 is then removed; (b) find the magnitude of the acceleration.

(a) Equilibrium: $F_1 + F_2 + F_3 = 0$

i: $-4 + 5 + a = 0$, so $a = -1$

j: $-4 + 12 + b = 0$, so $b = -8$

(b) The resultant is now $F_1 + F_3 = -F_2$, of magnitude $\sqrt{(5^2 + 12^2)} = 13$ N

$a = 13 \div 2$

Answer: (a) a = -1, b = -8 (b) 6.5 m s^{-2}

3 Surds: sums of fractions (AQA, Edexcel, OCR, MEI, Eduqas)

Show that $\frac{4}{\sqrt{5} - 2} + \frac{2}{\sqrt{5} + 2}$ can be written as $p + q\sqrt{5}$, and find p and q.

Common denominator $(\sqrt{5} - 2)(\sqrt{5} + 2) = 5 - 4 = 1$

Numerator $= 4(\sqrt{5} + 2) + 2(\sqrt{5} - 2)$

$= 4 + 6\sqrt{5}$

Divide by 1.

Answer: p = 4, q = 6

4 Newton's second law with vectors (AQA, Edexcel, OCR, MEI, Eduqas)

A particle of mass 3 kg is acted on by forces $F_1 = (7\mathbf{i} - 9\mathbf{j})$ N and F_2. Its acceleration is $(4\mathbf{i} + \mathbf{j})$ m s^{-2}. Find (a) F_2, (b) the angle between F_2 and **i**, to 1 d.p.

F = ma: $F_1 + F_2 = 3(4\mathbf{i} + \mathbf{j}) = 12\mathbf{i} + 3\mathbf{j}$

(a) $F_2 = (12\mathbf{i} + 3\mathbf{j}) - (7\mathbf{i} - 9\mathbf{j}) = 5\mathbf{i} + 12\mathbf{j}$

(b) $|F_2| = \sqrt{(5^2 + 12^2)} = 13$

$\cos\theta = 5 \div 13$, so $\theta = 67.4°$

Answer: (a) F_2 = (5i + 12j) N (b) 67.4°

5 Arrangements with restrictions (OCR, MEI, Eduqas, CIE)

How many arrangements of the letters of CHAMPION begin and end with a consonant?

The first letter: 5 consonants

The last letter: 4 consonants left

The middle 6 letters: 6! ways

Total $= 5 \times 4 \times 6! = 14400$

Answer: 14400

6 Circle touching an axis (diagram) (AQA, Edexcel, OCR, MEI, Eduqas)

The diagram shows a circle that touches the x-axis at A(−1, 0) and passes through B(2, 1).
Find the equation of the circle.

Touches the x-axis at A, so the centre is (−1, r) and the radius is r.

B on the circle: $(3)^2 + (1 - r)^2 = r^2$

$9 + 1 - 2r = 0$ so $r = 5$

Answer: $(x + 1)^2 + (y - 5)^2 = 25$

7 Normal approximation to the binomial (All boards)

X ~ B(80, 0.6). Use a normal approximation to estimate P(X ≥ 50) to 3 significant figures.

$np = 48$, $npq = 19.2$, so Y ~ N(48, 19.2)

With continuity correction: P(Y > 49.5)

$z = \frac{49.5 - 48}{4.38} = 0.342$, P(Z > 0.342) = 0.366

Answer: 0.366

8 Differential equations in context (All boards)

A drink at 102°C is left in a room at 22°C. Its temperature θ °C satisfies $\frac{d\theta}{dt} = -k(\theta - 22)$.

After 15 minutes the temperature is 67°C. Find k, then the time for it to reach 32°C, to 3 s.f.

$\int \frac{1}{\theta - 22}\, d\theta = \int -k\, dt$, so $\ln(\theta - 22) = -kt + C$

$t = 0$: $\theta - 22 = 80$, so $\theta - 22 = 80e^{-kt}$

$t = 15$: $45 = 80e^{-15k}$, so $k = \frac{\ln(\frac{16}{9})}{15} = 0.0384$

$10 = 80e^{-kt}$: $t = \frac{\ln(8)}{k} = 54.2$ minutes

Answer: k = 0.0384, t = 54.2 minutes

9 Exponential growth and decay modelling (All boards)

The population of a town is modelled by $P = 500\, e^{kt}$, where t is time in years.
After 6 years, P = 2500.
Find (a) the exact value of k, (b) the time for P to reach 5000, to 3 s.f.

$500\, e^{6k} = 2500$, so $e^{6k} = 5$ and $k = \frac{\ln 5}{6}$

k = 0.268 (3 s.f.)

$500\, e^{kt} = 5000$ gives $kt = \ln 10$, so $t = \frac{\ln 10}{k} = 8.58$ years

Answer: (a) $k = \frac{\ln 5}{6}$ (b) t = 8.58 years

10 Tangent conditions (All boards)

The line $y = kx - 1$ is a tangent to the curve $y = x^2 - 2x + 15$.
(a) Find the possible values of k.
(b) For the larger value of k, find the coordinates of the point of contact.

Equate: $x^2 - 2x + 15 = kx - 1$

$x^2 + (-2 - k)x + 16 = 0$

Tangent: discriminant = 0: $(-2 - k)^2 = 64$

$-2 - k = \pm 8$ so $k = 6$ or -10

$k = 6$: $x = \frac{k - (-2)}{2} = 4$, $y = 23$

Answer: (a) k = −10 or k = 6 (b) (4, 23)