Day 67 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Three-set Venn diagram counts (All boards)

Of 35 students, 12 study Maths (M), 16 Physics (P) and 16 Chemistry (C).
2 study M and P, 6 study M and C, 6 study P and C, and 1 study all three.
One student is chosen at random. Given that a student studies Chemistry, find the probability that they study all three.

Venn regions: M only 5, P only 9, C only 5,
MP only 1, MC only 5, PC only 5, all three 1, outside all 4
Studies Chemistry: 16
Studies all three: 1
Probability = $\frac{1}{16}$

Answer: $\mathbf{\frac{1}{16}}$

2 Completing the square (All boards)

$f(x) = -2x^2 + 20x - 46$.
(a) Write f(x) in the form $a(x + p)^2 + q$.
(b) State the maximum value of f(x) and the value of x at which it occurs.
(c) Solve $f(x) = 0$, giving surds in simplest form.

$f(x) = -2(x - 5)^2 + 4$
(b) maximum value 4 when $x = 5$
(c) $-2(x - 5)^2 = -4$, so $(x - 5)^2 = 2$
$x - 5 = \pm\sqrt{2}$
Answer: (a) $-2(x - 5)^2 + 4$ (b) maximum 4 at $x = 5$ (c) $x = 5 \pm \sqrt{2}$

3 Trigonometric equations (quadratic in sin or cos) (All boards)

Solve $2\sin^2\theta - 4\cos\theta + 4 = 0$
for $0° \le \theta \le 360°$.

Use $\sin^2\theta \equiv 1 - \cos^2\theta$: $2\cos^2\theta + 4\cos\theta - 6 = 0$
Solving the quadratic: $\cos\theta = 1$ or $\cos\theta = -3$
$\cos\theta = 1$: $\theta = 0°, 360°$
$\cos\theta = -3$ has no solutions ($|\cos\theta| \le 1$)
Answer: $\theta = 0°, 360°$

4 Inverse of a quadratic function (All boards)

$f(x) = 3x^2 - 6x + 9$, $x \ge 1$.
Find $f^{-1}(x)$ and state its domain.

$f(x) = 3(x - 1)^2 + 6$
Let $y = f(x)$: $(x - 1)^2 = \frac{y - 6}{3}$
$x \ge 1$, so take the positive root: $x = 1 + \sqrt{\frac{y - 6}{3}}$
Answer: $f^{-1}(x) = 1 + \sqrt{\frac{x - 6}{3}}$, $x \ge 6$

5 Projectile from and to the ground (AQA, Edexcel, OCR, MEI, Eduqas)

A ball is projected from a point O on horizontal ground at 17 m s^{-1} at angle α above the horizontal, $\tan\alpha = \frac{5}{12}$, and lands at A. Find (a) the time of flight, (b) the greatest height, (c) OA. Take g = 9.8. Give answers to 3 s.f. if not exact.

Horizontal component = $17\times\frac{12}{13} = \frac{204}{13}$; vertical component = $17\times\frac{5}{13} = \frac{85}{13}$

(a) Vertical: $0 = \frac{85}{13}t - 4.9t^2$, so $t = \frac{850}{637}$

(b) At the top: $0 = \left(\frac{85}{13}\right)^2 - 2\times9.8\times H$, so $H = \frac{36125}{16562}$

(c) OA = $\frac{204}{13}\times\frac{850}{637}$

Answer: (a) 1.33 s (b) 2.18 m (c) 20.9 m

6 Discrete random variable with unknown constant (Edexcel, OCR, MEI, Eduqas, CIE)

The random variable X has $P(X = x) = k(x^2 + 1)$ for x = 0, 1, 2, 3.

(a) Show that $k = \frac{1}{18}$. (b) Find E(X) and Var(X).

(c) Find $P(X \geq 1)$.

(a) $\Sigma P(X = x) = 1$: 1 + 2 + 5 + 10 = 18, so 18k = 1

(b) $E(X) = 1\times\frac{1}{9} + 2\times\frac{5}{18} + 3\times\frac{5}{9} = \frac{7}{3}$

$E(X^2) = \frac{56}{9}$, so $Var(X) = \frac{56}{9} - \left(\frac{7}{3}\right)^2 = \frac{7}{9}$

(c) $P(X \geq 1) = \frac{1}{9} + \frac{5}{18} + \frac{5}{9} = \frac{17}{18}$

Answer: (b) $E(X) = \frac{7}{3}$, $Var(X) = \frac{7}{9}$ (c) $\frac{17}{18}$

7 Binomial expansion: terms with negative powers of x (All boards)

Find the coefficient of x^8 in the expansion of

$\left(2x^3 - \frac{3}{x}\right)^8$.

General term: ${}^8C_r\,(2x^3)^{8-r}\,(-3x^{-1})^r$

Power of x: $3(8 - r) - 1r = 8$

$24 - 4r = 8$ so r = 4

Coefficient = ${}^8C_4\times2^4\times(-3)^4 = 90720$

Answer: 90720

8 Differential equations in context (All boards)

A drink at 75°C is left in a room at 15°C. Its temperature θ °C satisfies $\frac{d\theta}{dt} = -k(\theta - 15)$.

After 15 minutes the temperature is 55°C. Find k, then the time for it to reach 25°C, to 3 s.f.

$\int\frac{1}{\theta - 15}\,d\theta = \int -k\,dt$, so $\ln(\theta - 15) = -kt + C$

t = 0: $\theta - 15 = 60$, so $\theta - 15 = 60e^{-kt}$

t = 15: $40 = 60e^{-15k}$, so $k = \frac{\ln\left(\frac{3}{2}\right)}{15} = 0.0270$

$10 = 60e^{-kt}$: $t = \frac{\ln(6)}{k} = 66.3$ minutes

Answer: k = 0.0270, t = 66.3 minutes

9 Ships on collision courses (vectors) (AQA, Edexcel, OCR, MEI, Eduqas)

At 12:00 ship P has position $(-\mathbf{i} + 3\mathbf{j})$ km and velocity $(3\mathbf{i} + \mathbf{j})$ km h^{-1}; ship Q has position $(-7\mathbf{i} - 9\mathbf{j})$ km and velocity $(5\mathbf{i} + 5\mathbf{j})$ km h^{-1}. Show that the ships collide and find the time and position vector of the collision.

At time t hours: $\mathbf{r}_P = (-\mathbf{i} + 3\mathbf{j}) + t(3\mathbf{i} + \mathbf{j})$, $\mathbf{r}_Q = (-7\mathbf{i} - 9\mathbf{j}) + t(5\mathbf{i} + 5\mathbf{j})$
Equal when: $\mathbf{i}$: $-1 + 3t = -7 + 5t$, so $t = 3$
$\mathbf{j}$: $3 + 3\times1 = 6$
Check with Q: $\mathbf{i}$: $-7 + 3\times5 = 8$, $\mathbf{j}$: $-9 + 3\times5 = 6$
Both components agree at $t = 3$, so they collide at $(8\mathbf{i} + 6\mathbf{j})$ km
Answer: t = 3 hours (at 15:00); r = (8i + 6j) km

10 Intersecting or skew lines (CIE (9709); Further Maths on other boards)

$$l_1: \mathbf{r} = \begin{pmatrix} 12 \\ -3 \\ 8 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 2 \\ -3 \end{pmatrix}$$

$$l_2: \mathbf{r} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$$

Determine whether l_1 and l_2 intersect. If they do, find the point of intersection.

Equate components: $12 - 3\lambda = 3 - 1\mu$; $-3 + 2\lambda = 3 + 3\mu$; $8 - 3\lambda = -1 + 3\mu$
Solving: $\lambda = 3$, $\mu = 0$, and all three equations are satisfied
Point of intersection = $(3, 3, -1)$
Answer: Intersect at (3, 3, −1)