Day 63 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Logarithmic equations (All boards)

Solve $\log_{10}(x + 6) + \log_{10}(x - 3) = 1$.

Combine: $\log_{10}[(x + 6)(x - 3)] = 1$, so $(x + 6)(x - 3) = 10^1 = 10$

$x^2 + 3x - 28 = 0$: $(x - 4)(x + 7) = 0$

$x = 4$ or $x = -7$; $x = -7$ makes $x - 3$ negative, so it is rejected

Answer: $x = 4$

2 Lift and cable tension (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A lift of mass 300 kg carries a person of mass 60 kg. The cable tension is 3456 N and the lift is accelerating. Find (a) the magnitude and direction of the acceleration, (b) the force of the floor on the person. Take $g = 9.8$.

Whole system (mass 360 kg), taking upwards as positive: $T - 360\,g = 360\,a$

$3456 - 3528 = 360\,a$, so $a = -0.2$

(a) a is negative, so the acceleration is 0.2 m s^{-2} downwards

(b) Person: $R - 60\,g = 60\,a$, so $R = 60(9.8 - 0.2)$

Answer: (a) 0.2 m s^{-2} downwards (b) 576 N

3 Binomial expansion: terms with negative powers of x (All boards)

Find the coefficient of x^{-13} in the expansion of

$(3x^3 + \frac{2}{x^2})^9$.

General term: ${}^9C_r\,(3x^3)^{9-r}\,(2x^{-2})^r$

Power of x: $3(9 - r) - 2r = -13$

$27 - 5r = -13$ so $r = 8$

Coefficient = ${}^9C_8 \times 3^1 \times 2^8 = 6912$

Answer: 6912

4 Scalar product, magnitude of a sum (CIE (9709); Further Maths on other boards)

$|a| = 8$, $|b| = 5$ and the angle between a and b is 135°.

Find the exact values of $a \cdot b$ and $|a + b|$.

$a \cdot b = |a||b|\cos 135° = 8 \times 5 \times -\frac{\sqrt{2}}{2} = -20\sqrt{2}$

$|a + b|^2 = |a|^2 + 2a \cdot b + |b|^2 = 64 + -40\sqrt{2} + 25 = 89 - 40\sqrt{2}$

$|a + b| = \sqrt{89 - 40\sqrt{2}}$

Answer: $\mathbf{a \cdot b = -20\sqrt{2}}$, $\mathbf{|a + b| = \sqrt{89 - 40\sqrt{2}}}$

5 Inverse of a quadratic function (All boards)

$f(x) = x^2 + 6x + 4$, $x \geq -3$.

Find $f^{-1}(x)$ and state its domain.

$f(x) = (x + 3)^2 - 5$

Let $y = f(x)$: $(x + 3)^2 = y + 5$

$x \geq -3$, so take the positive root: $x = -3 + \sqrt{y + 5}$

Answer: $\mathbf{f^{-1}(x) = -3 + \sqrt{x + 5}}$, $\mathbf{x \geq -5}$

6 Addition formulae with given sin and cos (All boards)

A is acute and B is obtuse, with $\sin A = \frac{8}{17}$ and $\cos B = -\frac{7}{25}$.

Find the exact values of cos(A+B) and cos(A−B).

$\cos A = \frac{15}{17}$ (A acute), $\sin B = \frac{24}{25}$ (B obtuse), so $\tan A = \frac{8}{15}$, $\tan B = -\frac{24}{7}$

$\cos(A+B) = \cos A \cos B - \sin A \sin B = \frac{15}{17} \times (-\frac{7}{25}) - \frac{8}{17} \times \frac{24}{25} = -\frac{297}{425}$

$\cos(A-B) = \cos A \cos B + \sin A \sin B = \frac{15}{17} \times (-\frac{7}{25}) + \frac{8}{17} \times \frac{24}{25} = \frac{87}{425}$

Answer: $\cos(A+B) = -\frac{297}{425}$, $\cos(A-B) = \frac{87}{425}$

7 Partial fractions: repeated factor (All boards)

Express $\frac{-7x^2 - 19x - 6}{(x+3)(x+1)^2}$ in the form $\frac{A}{(x+3)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$, and find A, B and C.

$-7x^2 - 19x - 6 \equiv A(x+1)^2 + B(x+3)(x+1) + C(x+3)$

$x = -3$: $-12 = A(-2)^2$ so $A = -3$

$x = -1$: $6 = C(2)$ so $C = 3$

Coefficient of x^2: $-7 = A + B$ so $B = -4$

Answer: $A = -3$, $B = -4$, $C = 3$

8 Binomial in two stages (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Eggs are defective with probability 0.25, independently. A tray of 10 eggs is rejected if it has at least 3 defective. From 8 such trays, find the probability that exactly 2 are rejected (3 s.f.).

$Y \sim B(10, 0.25)$. $P(\text{rejected}) = P(Y \geq 3) = 1 - 0.526 = 0.474$

$W \sim B(8, 0.474)$ is the number rejected

$P(W = 2) = C(8, 2)(0.474)^2(1 - 0.474)^6 = 0.133$

Answer: 0.133

9 Type I and Type II errors (AQA, Edexcel, OCR, MEI, Eduqas)

$X \sim B(38, p)$. H_0: $p = 0.5$ is tested against H_1: $p < 0.5$ using a 5% significance level.

(a) Find the critical region. (b) State the probability of a Type I error.

(c) Find the probability of a Type II error when $p = 0.4$ (3 s.f.).

(a) Largest/smallest region with probability ≤ 0.05 under H_0: $X \leq 13$

(b) Type I error = $P(\text{reject } H_0 \mid p = 0.5) = 0.0365$

(c) Type II error = P(not in critical region | $p = 0.4$)

$P(X \geq 14)$ when $p = 0.4$: 0.710

Answer: (a) $X \leq 13$ (b) 0.0365 (c) 0.710

10 Closest approach of two particles (Edexcel, OCR, MEI)

At t = 0, P and Q have position vectors $(9\mathbf{i} + 13\mathbf{j})$ m and $(-4\mathbf{i} + 4\mathbf{j})$ m and velocities $(-6\mathbf{i} + 5\mathbf{j})$ m s^{-1} and $(-2\mathbf{i} + 2\mathbf{j})$ m s^{-1}. Find (a) the time when they are closest, (b) the least distance between them.

$PQ = r_P - r_Q = (13\mathbf{i} + 9\mathbf{j}) + t(-4\mathbf{i} + 3\mathbf{j})$

$d^2 = (-4t + 13)^2 + (3t + 9)^2 = 25t^2 - 50t + 250$

(a) $d(d^2)/dt = 0$: $50t + -50 = 0$, so $t = 1$

(b) At $t = 1$: $PQ = 9\mathbf{i} + 12\mathbf{j}$, so $d = \sqrt{225}$

Answer: (a) t = 1 s (b) 15 m