Day 58 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Geometric series: smallest n (All boards)

A geometric series has first term 3 and common ratio $\frac{3}{2}$.

Find the smallest n for which the sum of the first n terms exceeds 1000.

$S_n = \frac{3(\frac{3}{2}^n - 1)}{(\frac{3}{2} - 1)} > 1000$

$\frac{3}{2}^n > \frac{503}{3}$

$n > \frac{\log \frac{503}{3}}{\log \frac{3}{2}} = 12.63$

Check: $S_{12} = 772.5$, $S_{13} = 1161.7$

Answer: n = 13

2 Constant acceleration (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A cyclist passes P with speed 5 m s^{-1} and accelerates uniformly. She travels 120 m in the next 15 s, reaching Q. Find (a) her acceleration, (b) her speed at Q, (c) her speed at the midpoint of PQ. Give answers to 3 s.f. if not exact.

(a) $s = ut + \frac{1}{2}at^2$: $120 = 5\times15 + \frac{1}{2}\times a\times15^2$, so $a = 0.4$

(b) $v = u + at = 5 + 0.4\times15 = 11$

(c) For the midpoint, $s = 60$: $v^2 = u^2 + 2as = 5^2 + 2\times0.4\times60$

$v^2 = 73$

Answer: (a) 0.4 m s^{-2} (b) 11 m s^{-1} (c) 8.54 m s^{-1}

3 Symmetric interval about the mean (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X, the length (mm), is N(78, 2^2).

Find the symmetric limits about the mean between which the central 95% of values lie (3 s.f.).

Each tail has area 0.025, so $P(Z < z) = 0.975$

$z = 1.96$

Limits: $78 \pm 1.96 \times 2$

$= 74.1$ and 81.9

Answer: 74.1 to 81.9

4 Connected rates of change: spheres, circles and cubes (AQA, Edexcel, OCR, MEI, Eduqas)

A circular ripple has area A cm^2 increasing at a constant 2 cm^2s^{-1}.

Find the rate at which the circumference is increasing when the radius is 10 cm.

$A = \pi r^2$, so $\frac{dA}{dr} = 2\pi r$ and $\frac{dr}{dt} = \frac{2}{2\pi r}$

$C = 2\pi r$, so $\frac{dC}{dt} = 2\pi \times \frac{dr}{dt} = 2\pi \times \frac{2}{2\pi \times 10} = \frac{1}{5}$

Answer: $\frac{1}{5}$ cm/s

5 Range of a composite function (All boards)

$f(x) = x^2 + 2x + 6$, $x \in \mathbb{R}$, and $g(x) = \frac{4}{x+1}$, $x > -1$.

(a) Find the range of f.

(b) Show that gf is defined and find its range.

$f(x) = (x + 1)^2 + 5 \geq 5$

So $f(x) \geq 5 > -1$, which lies in the domain of g, so gf is defined.

$gf(x) = \frac{4}{f(x)+1}$, with $f(x) \geq 5$, so $f(x) + 1 \geq 6$

Largest value is $\frac{4}{6} = \frac{2}{3}$ at the minimum of f; $gf(x) > 0$.

Answer: (a) $f(x) \geq 5$ (b) $0 < gf(x) \leq \frac{2}{3}$

6 Two-tailed binomial test critical region (AQA, Edexcel, OCR, MEI, Eduqas)

$X \sim B(23, p)$. H_0: $p = 0.4$ is tested against H_1: $p \neq 0.4$ at the 5% level.

(a) Find the critical region.

(b) Find the actual significance level of the test (3 s.f.).

(a) Each tail ≤ 0.025. $P(X \leq 4) = 0.0190$, $P(X \leq 5) = 0.0540$

$P(X \geq 15) = 0.0128$, $P(X \geq 14) = 0.0349$

Critical region: $X \leq 4$ or $X \geq 15$

(b) Actual level = 0.0190 + 0.0128 = 0.0318

Answer: (a) $X \leq 4$ or $X \geq 15$ (b) 0.0318

7 Perpendicular vectors and a constant (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The vectors $\begin{pmatrix} k \\ k \\ -2 \end{pmatrix}$ and $\begin{pmatrix} k \\ 3 \\ 5 \end{pmatrix}$ are perpendicular.

Find the possible values of k and, for the larger value, the exact length of the first vector.

Scalar product = 0: $k^2 + 3k - 10 = 0$

$(k + 5)(k - 2) = 0$

$k = -5$ or $k = 2$. For $k = 2$ the first vector is $\begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$, with length $\sqrt{12} = 2\sqrt{3}$

Answer: $k = -5$ or 2; length $2\sqrt{3}$

8 Exponential equations: quadratic in b^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $2^{2x} - 10 \times 2^x + 16 = 0$.

Let $y = 2^x$. Then $y^2 - 10y + 16 = 0$

$(y - 2)(y - 8) = 0$

$y = 2$ or $y = 8$

$2^x = 2$ gives $x = 1$; $2^x = 8$ gives $x = 3$

Answer: $x = 1$ or $x = 3$

9 Particles on two smooth inclined planes (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

The system is released from rest. Find (a) the magnitude of the acceleration and which particle moves down, (b) the tension. Take $g = 9.8$. Give answers to 3 s.f. if not exact.

A: weight component along plane = $5g \sin 30°$; B: $6g \sin 45°$

B moves down, A up ($6 \sin 45° > 5 \sin 30°$): F = ma for the system

$6g \sin 45° - 5g \sin 30° = (5 + 6)a$

(a) $a = 1.55$

(b) For A: $T - 5g \sin 30° = 5a$, so $T = 32.3$

Answer: (a) 1.55 m s^{-2}, B moves down (b) 32.3 N

10 Volumes of revolution (CIE (9709); Further Maths on other boards)

The region bounded by the curve $y = x^2 + 1$, the x-axis, $x = 0$ and $x = 1$ is rotated through 360° about the x-axis. Find the exact volume of the solid formed.

$y^2 = (x^2 + 1)^2 = x^4 + 2x^2 + 1$, so $V = \pi \int_0^1 y^2 \, dx$

$= \pi \times \frac{28}{15}$

$V = \frac{28\pi}{15}$

Answer: $\frac{28\pi}{15}$