Day 49 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Exact values from the addition formulae (All boards)

By writing 105° as 60° + 45°, find the exact value of

cos 105°, in the form $\frac{\sqrt{a} \pm \sqrt{b}}{c}$.

$\cos 105° = \cos(60° + 45°) = \cos 60° \cos 45° - \sin 60° \sin 45°$

$= \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2}$

$= \frac{\sqrt{2} - \sqrt{6}}{4}$

Answer: $\mathbf{\frac{\sqrt{2} - \sqrt{6}}{4}}$

2 Conservation of momentum (CIE)

Particles A (2 kg) and B (7 kg) move in the same direction on a smooth horizontal line at 5 m s^{-1} and 5 m s^{-1}. They collide and coalesce. Find (a) the speed of the combined particle, (b) the kinetic energy lost in the collision.

Conservation of momentum (A's direction positive):
$2\times5 + 7\times5 = 9v$, so $v = 5$
KE before = $\frac{1}{2}\times2\times5^2 + \frac{1}{2}\times7\times5^2 = 112.5$
KE after = $\frac{1}{2}\times9\times5^2 = 112.5$
Loss = $112.5 - 112.5$

Answer: (a) 5 m s^{-1} (b) 0 J

3 Index equations solved with logarithms (All boards)

Solve $3^{3x} = 7^{3x+3}$.

Give your answer to 3 significant figures.

Take logs: $3x \ln 3 = (3x + 3) \ln 7$
Collect x terms: $x(3 \ln 3 - 3 \ln 7) = 3 \ln 7$

$x = \frac{3 \ln 7}{3 \ln 3 - 3 \ln 7}$

$x = -2.30$

Answer: x = −2.30

4 Binomial mean and variance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X has a binomial distribution with mean 12 and variance 4.8.

(a) Find n and p.

(b) Find P(X = 11) to 3 significant figures.

(a) $np = 12$ and $npq = 4.8$

$q = \frac{npq}{np} = \frac{4.8}{12} = 0.4$, so $p = 0.6$

$n = \frac{12}{0.6} = 20$

(b) $P(X = 11) = C(20, 11)(0.6)^{11}(0.4)^9 = 0.160$

Answer: (a) n = 20, p = 0.6 (b) 0.160

5 Modulus inequalities (All boards)

Solve $|2x + 3| < |x - 4|$.

Square both sides: $(2x + 3)^2 < (x - 4)^2$
$((2x + 3) - (x - 4))((2x + 3) + (x - 4)) < 0$
$(x + 7)(3x - 1) < 0$

Critical values $x = -7$ and $x = \frac{1}{3}$; sketch the (positive or negative) quadratic.

Answer: $\mathbf{-7 < x < \frac{1}{3}}$

6 Friction: find μ then stopping distance (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 15 kg box is pulled from rest along rough horizontal ground by a horizontal force of 53.1 N and moves 10.8 m in 6 s. Find (a) the coefficient of friction. The force is then removed; (b) find the further distance moved. Take g = 9.8. Give answers to 3 s.f. if not exact.

$s = \frac{1}{2}at^2$: $10.8 = \frac{1}{2}a \times 6^2$, so $a = 0.6$

(a) $P - \mu 15\,g = 15\,a$: $53.1 - 147\,\mu = 9$, so $\mu = 0.3$

(b) $v = at = 0.6 \times 6 = 3.6$; after removal deceleration $= \mu g = 2.94$

$0 = 3.6^2 - 2 \times 2.94 \times s$, $s = 2.20$

Answer: (a) $\mu = 0.3$ (b) 2.20 m

7 Differential equations in context (All boards)

Water drains from a tank so that the depth h metres falls at a rate proportional to $\sqrt{h}$. Initially h = 9, and after 6 hours h = 4. Form and solve a differential equation to find how long, in total, the tank takes to empty.

$\frac{dh}{dt} = -k\sqrt{h}$: $\int h^{-\frac{1}{2}}\,dh = \int -k\,dt$, so $2\sqrt{h} = -kt + C$

$t = 0$: $C = 2 \times 3 = 6$, so $2\sqrt{h} = 6 - kt$

$t = 6$, $h = 4$: $4 = 6 - 6k$, so $k = \frac{1}{3}$

Empty when $h = 0$: $kt = 6$, so $t = \frac{6}{\frac{1}{3}} = 18$ hours

Answer: 18 hours

8 Geometric sequences: finding a and r (All boards)

The 2nd term of a geometric sequence is −12 and the 4th term is −108.
Find the common ratio and the first term.
Find the sum of the first 8 terms.

$r^2 = \frac{-108}{-12} = 9$ so $r = -3$

$a\,r^1 = -12$ so $a = 4$

$S_8 = \frac{a(r^8 - 1)}{(r - 1)} = \frac{4(6561 - 1)}{-4}$

Answer: $r = -3$, $a = 4$, $S_8 = -6560$

9 Recurrence relations: limits (All boards)

$u_1 = 8$ and $u_{n+1} = \frac{1}{4}u_n + 3$.

(a) Find the limit L of the sequence as $n \to \infty$.
(b) Find the least n for which $|u_n - L| < 0.1$.

(a) $L = \frac{1}{4}L + 3$ gives $L = \frac{3}{1 - \frac{1}{4}} = 4$

(b) $u_n - L = (u_1 - L)\,r^{n-1} = 4 \times \left(\frac{1}{4}\right)^{n-1}$

$|4| \times \left(\frac{1}{4}\right)^{n-1} < \frac{1}{10}$

Check: n = 3 gives 0.25000, n = 4 gives 0.06250

Answer: (a) L = 4 (b) n = 4

10 Type I and Type II errors (AQA, Edexcel, OCR, MEI, Eduqas)

$X \sim B(26, p)$. H_0: $p = 0.4$ is tested against H_1: $p > 0.4$ using a 5% significance level.

(a) Find the critical region. (b) State the probability of a Type I error.

(c) Find the probability of a Type II error when $p = 0.6$ (3 s.f.).

(a) Largest/smallest region with probability ≤ 0.05 under H_0: $X \geq 16$

(b) Type I error = P(reject H_0 | $p = 0.4$) = 0.0217

(c) Type II error = P(not in critical region | $p = 0.6$)

$P(X \leq 15)$ when $p = 0.6$: 0.479

Answer: (a) $X \geq 16$ (b) 0.0217 (c) 0.479