Day 48 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Logarithmic equations (All boards)

Solve $2\log_2 x - \log_2(x + 4) = 1$.

$\log_2 \frac{x^2}{x+4} = 1$, so $\frac{x^2}{x+4} = 2^1 = 2$

$x^2 = 2(x + 4)$: $x^2 - 2x - 8 = 0$

$x = 4$ or $x = -2$; $\log_2 x$ needs $x > 0$, so reject the negative root

Answer: $x = 4$

2 Exponential equations: quadratic in b^x (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Solve $3 \times 3^{2x} - 12 \times 3^x + 9 = 0$.

Let $y = 3^x$. Then $3y^2 - 12y + 9 = 0$

$3(y - 3)(y - 1) = 0$

$y = 3$ or $y = 1$

$3^x = 3$ gives $x = 1$; $3^x = 1$ gives $x = 0$

Answer: $x = 0$ or $x = 1$

3 Equilibrium on a slope: range of force (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A 2 kg particle rests on a rough plane inclined at α, $\tan\alpha = \frac{24}{7}$, $\mu = \frac{1}{3}$. A force of P N acts up a line of greatest slope and the particle stays at rest. Find the range of values of P. Take $g = 9.8$. Give answers to 3 s.f. if not exact.

$\sin\alpha = 0.96$, $\cos\alpha = 0.28$, $R = 2g\cos\alpha = 5.488$ N

Limiting friction $= \mu R = \frac{686}{375}$ N

P smallest: friction acts up the slope: $P + \mu R = 2g\sin\alpha$, $P = 18.816 - \frac{686}{375}$

P greatest: friction acts down the slope: $P = 18.816 + \frac{686}{375}$

Answer: $17.0\text{ N} \leq P \leq 20.6\text{ N}$

4 Poisson probabilities (CIE)

Calls reach a switchboard at a mean rate of 3 per minute, as a Poisson process.
Find the probability that in one such period there are
(a) exactly 2 (b) at least 2 (c) fewer than 2 (3 s.f.).

$X \sim Po(3)$

(a) $P(X = 2) = e^{-3}\, 3^2/2! = 0.224$

(b) $1 - P(X \leq 1) = 1 - 0.199 = 0.801$

(c) $P(X \leq 1) = 0.199$

Answer: (a) 0.224 (b) 0.801 (c) 0.199

5 Modulus graph and a line (All boards)

The diagram shows the graphs of $y = |x + 1|$ and $y = \frac{1}{2}x + 2$.

The graphs meet at A and B.
Find the coordinates of A and B.

$|x + 1| = \frac{1}{2}x + 2$

Solve $x + 1 = \pm(\frac{1}{2}x + 2)$.

$x = -2$ and $x = 2$ (both checked in the original equation)

$y = 1$ and $y = 3$

Answer: A(−2, 1), B(2, 3)

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6 Parametric differentiation: tangent and normal (All boards)

A curve has parametric equations $x = t^2 - 1$, $y = t^3 - 3t$.

Find $\frac{dy}{dx}$ in terms of t, and the equation of the normal at the point where t = 2, in the form ax + by + c = 0 (integers).

$\frac{dy}{dx} = \frac{dy}{dt \div dx/dt} = \frac{3t^2 - 3}{2t}$

t = 2: point (3, 2), gradient $\frac{9}{4}$

Normal gradient = $-1 \div \frac{9}{4} = -\frac{4}{9}$, so normal: $4x + 9y - 30 = 0$

Answer: $\frac{dy}{dx} = \frac{3t^2 - 3}{2t}$; $4x + 9y - 30 = 0$

7 Conditional binomial probability (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(20, 0.6). Find $P(X = 6 \mid X \geq 4)$ to 3 significant figures.

$P(X = 6 \mid X \geq 4) = \frac{P(X = 6)}{P(X \geq 4)}$ (since X = 6 implies $X \geq 4$)

$P(X = 6) = 0.00485$

$P(X \geq 4) = 1 - P(X \leq 3) = 1.00$

Ratio = 0.00485

Answer: 0.00485

8 Range of a composite function (All boards)

$f(x) = x^2 + 6x + 12$, $x \in \mathbb{R}$, and $g(x) = \frac{4}{x + 1}$, $x > -1$.

(a) Find the range of f.

(b) Show that gf is defined and find its range.

$f(x) = (x + 3)^2 + 3 \geq 3$

So $f(x) \geq 3 > -1$, which lies in the domain of g, so gf is defined.

$gf(x) = \frac{4}{f(x) + 1}$, with $f(x) \geq 3$, so $f(x) + 1 \geq 4$

Largest value is $\frac{4}{4} = 1$ at the minimum of f; $gf(x) > 0$.

Answer: (a) $f(x) \geq 3$ (b) $0 < gf(x) \leq 1$

9 Differential equations in context (All boards)

Water drains from a tank so that the depth h metres falls at a rate proportional to $\sqrt{h}$. Initially h = 49, and after 5 hours h = 9.

Form and solve a differential equation to find how long, in total, the tank takes to empty.

$\frac{dh}{dt} = -k\sqrt{h}$: $\int h^{-\frac{1}{2}}\, dh = \int -k\, dt$, so $2\sqrt{h} = -kt + C$

t = 0: $C = 2 \times 7 = 14$, so $2\sqrt{h} = 14 - kt$

t = 5, h = 9: $6 = 14 - 5k$, so $k = \frac{8}{5}$

Empty when h = 0: $kt = 14$, so $t = \frac{14}{\frac{8}{5}} = \frac{35}{4}$ hours

Answer: $\frac{35}{4}$ hours

10 Work-energy on a rough slope (CIE)

A 5 kg block is released from rest on a rough plane inclined at α, $\tan\alpha = \frac{3}{4}$, $\mu = 0.25$, and slides 2 m down a line of greatest slope. Use work and energy to find (a) the work done against friction, (b) its speed after 2 m. Take $g = 10$.

$R = 5g\cos\alpha = 40$ N, $F = \mu R = 10$ N

(a) Work done against friction $= F \times d = 10 \times 2$

(b) Loss of PE $= 5g \times 2\sin\alpha = 60$ J

Work-energy: $\frac{1}{2}\times 5 \times v^2 = 60 - 20 = 40$

$v^2 = 16$

Answer: (a) 20 J (b) 4 m s^{-1}