Day 45 **A Level Maths: A Tier 10-a-day: Model Solutions**

1 Proof by deduction (AQA, Edexcel, OCR, MEI, Eduqas)

Prove that $(n + 3)^2 - (n - 3)^2$ is a multiple of 12 for every integer n.

Difference of two squares:

$(n + 3)^2 - (n - 3)^2 = (n + 3 - n + 3)(n + 3 + n - 3)$

$= 6 \times 2n = 12n$

12n is a multiple of 12.

Answer: 12n, a multiple of 12

2 Smallest whole number solving an exponential inequality (All boards)

Find the least integer n for which $5^n > 10^5$.

$n \lg 5 > 5$, so $n > \frac{5}{\lg 5} = 7.15$

Since n must be an integer, n = 8

Answer: n = 8

3 Binomial probabilities (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

X ~ B(24, 0.2). Find, to 3 significant figures:

(a) P(X = 7) (b) P(X ≥ 7) (c) P(X ≤ 6)

(a) $C(24, 7)(0.2)^7(0.8)^{17} = 0.0998$

(b) $1 - P(X \leq 6) = 1 - 0.811 = 0.189$

(c) $P(X \leq 6) = 0.811$

Answer: (a) 0.0998 (b) 0.189 (c) 0.811

4 Beam about to tilt (AQA, Edexcel, OCR, MEI)

A uniform plank AB (mass 10 kg, length 4 m) rests horizontally on supports at C and D, where AC = 0.5 m and DB = 1 m. A 10 kg particle is placed at E, AE = x m, and the plank is about to tilt about D. Find x.

About to tilt about D, so the reaction at C is zero and E lies between D and B.

Moments about D: $10\text{ g} \times (x - 3) = 10\text{ g} \times (3 - 2)$

$10(x - 3) = 10$

$x = 3 + 1$

Answer: x = 4

5 Factor theorem: roots with surds (All boards)

$f(x) = 3x^3 - 30x - 9$.

(a) Show that (x + 3) is a factor of f(x).

(b) Solve f(x) = 0, giving surds in simplest form.

$f(-3) = -81 + 90 - 9 = 0$, so (x + 3) is a factor.

$f(x) = 3(x + 3)(x^2 - 3x - 1)$

$x^2 - 3x - 1 = 0$ gives $x = \frac{3 \pm \sqrt{13}}{2}$

Answer: $x = -3$ or $x = \frac{3 \pm \sqrt{13}}{2}$

6 Small angle approximations: solving an equation (AQA, Edexcel, OCR, MEI, Eduqas)

θ is small and measured in radians. Use small angle approximations to show that

$4\sin 2\theta + 4\cos 3\theta = 1.5$

can be written as $a\theta^2 + b\theta + c = 0$ with integers a, b, c. Hence find the approximate solution with $0 < \theta < 0.8$, to 3 s.f.

$\sin 2\theta \approx 2\theta$, $\cos 3\theta \approx 1 - \frac{9}{2}\theta^2$

$4 \times 2\theta + 4(1 - \frac{9}{2}\theta^2) = 1.5$

Multiply by 2 and rearrange: $36\theta^2 - 16\theta - 5 = 0$

$\theta = \frac{16 \pm \sqrt{976}}{72}$, so $\theta = 0.656$

Answer: a = 36, b = −16, c = −5; $\theta \approx 0.656$

7 Constant speed against friction (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

A crate of mass 12 kg is pulled at constant speed along rough horizontal ground, $\mu = 0.3$, by a force of P N at angle θ above the horizontal, $\tan\theta = \frac{3}{4}$. Find P and the normal reaction. Take g = 9.8.

Constant speed, so the resultant force is zero. $\sin\theta = 0.6$, $\cos\theta = 0.8$
Vertical: $R = 12g - P\sin\theta$
Horizontal: $P\cos\theta = \mu R = 0.3(12g - P\sin\theta)$
$P(0.8 + 0.18) = 0.3 \times 12g$
$P = 36.0$, $R = 96.0$
Answer: P = 36 N, R = 96 N

8 Binomial in two stages (AQA, Edexcel, OCR, MEI, Eduqas, CIE)

Bulbs are defective with probability 0.2, independently. A pack of 10 bulbs is rejected if it has at least 3 defective. From 12 such packs, find the probability that exactly 3 are rejected (3 s.f.).

$Y \sim B(10, 0.2)$. $P(\text{rejected}) = P(Y \geq 3) = 1 - 0.678 = 0.322$
$W \sim B(12, 0.322)$ is the number rejected
$P(W = 3) = C(12, 3)(0.322)^3(1 - 0.322)^9 = 0.222$
Answer: 0.222

9 Circle touching both axes (AQA, Edexcel, OCR, MEI, Eduqas)

A circle touches both coordinate axes and passes through the point (9, −2). Its centre lies in the fourth quadrant. Find the equations of the two possible circles.

Centre $(r, -r)$, radius $r > 0$.
Through (9, −2): $(9 - r)^2 + (-2 + r)^2 = r^2$
$r^2 - 22r + 85 = 0$
$(r - 5)(r - 17) = 0$, so $r = 5$ or 17
Answer: $(x - 5)^2 + (y + 5)^2 = 25$ or $(x - 17)^2 + (y + 17)^2 = 289$

10 Intersecting or skew lines (CIE (9709); Further Maths on other boards)

l_1: $\mathbf{r} = \begin{pmatrix}2\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}-3\\2\\3\end{pmatrix}$

l_2: $\mathbf{r} = \begin{pmatrix}-4\\3\\1\end{pmatrix} + \mu\begin{pmatrix}-3\\-3\\-2\end{pmatrix}$

Determine whether l_1 and l_2 intersect. If they do, find the point of intersection.

Equate components: $2 - 3\lambda = -4 - 3\mu$; $4 + 2\lambda = 3 - 3\mu$; $0 + 3\lambda = 1 - 2\mu$
Solving: $\lambda = 1$, $\mu = -1$, and all three equations are satisfied
Point of intersection = (−1, 6, 3)
Answer: Intersect at (−1, 6, 3)